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The algorithmic music and the Musical Offering of J.S.Bach

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ABSTRACT

The object of this work is an algebraic – geometric analysis of the Canons of the Musical Offering of J.S.Bach. Here is built a simulation model, thanks to the logic of imitation and thanks to the classical techniques of the Harmony, which are present in the Musical Offering. Many of these techniques give only musical criteria to understand, compose and classify Canons, so we have translated them in the mathematical language, using the affinities of the euclidean bidimensional space. More generally a similar approach allows to write each canon of each period in a suitable linear system. A treatment of this kind for a musical piece offers a new method to compose, giving different numerical values to the coefficients of the linear system.

Keywords: *Computational Musical Models* .

INTRODUCTION

During the evolution phase there is a moment of the education, between 15 and 20 years old, where it is privileged the rational component. This spirit leads to ask reasons why some musical phenomena happen, so one is often subject to questions on fundamental musical troubles. These troubles can be better solved using the reasoning. A lot of people, even who has already received a formation in Music, is surprised to find a rational sense in problems which they have treated for a long time under a sentimental outline. Surely it is possible to introduce the reasoning in many musical phenomena, but the different inclination of anyone and the variety of the existing problems lead to a sectorial approach.

A very attractive trend is suggested by Harmony and Contrapunctus. There are chords which, played in a certain way, induce a tonality, but played in another way they do not induce the same tonality and these are examples of Symmetries in Music. Little trivial things which can be better explained when we make a suitable discourse and the mathematical language is particularly right for similar circumstances.

The *Musical Offering* is not interesting by itself, instead, a phylological analysis of the Contrapunctus, which here is used, makes it a very nice piece and its beautifulness is bigger thanks to the precise scientific mentality of the composition. Under this point of view some works are self-presented; further than Musical Offering of J.S.Bach, there is also his *Art of Fugue*.

A large literature exists about J.S.Bach and Mathematics (see Assayag & others (2002), Benson (2003), Fauvel & others (2003), Fergola (2002), Galante (1998), Hofstadter (2001), Scimemi (1983,1992,1994,1997,2002), Tannery (1902)), since his production is characterized by deep sense of scheme. These qualities are clear for a musician: they constitutes a real structure in the composition of the pieces. To understand these structures there are some subjects, i.e. Harmony, which give only musical criteria. A corresponding criterion in mathematical language is offered by the affinities of the euclidean bidimensional

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space, so we are able to start an algebraic – geometric treatment of he Musical Offering.

The present work is subdivided in five sections and it describes a simulation model which can be extended to other musical compositions, which are not only bachian Canons. More generally such model is right for compositions where it is used a logic of *imitation*. Beginning with easy Canons in 2 voices, where the imitation happens only between two instruments, we get to canons in 3 or 4 voices, where the melodic interweaving is more complicated. Section 2 is devoted to formalize a Canon, using linear systems (Equation **(2.4)**) and this classifies a Canon thanks to the euclidean bidimensional affinity which is associated (Theorem 3.1). Before getting to Equation **(2.4)**, we need to introduce the time–frequency diagrams (Section 1), which show the temporal evolution of the instruments (voices) involved in the Canon and we adopt the equal temperament (see Basso (1985), Frova (1990), Schweitzer (1967)). On the other hand (Section 5) a similar treatment allows a method to compose, giving different numerical values at the coefficients of the linear system **(5.1)**. This way to compose shows a first historical example of *Algorithmic Music* and many contemporanean pieces are often characterized by a logic of this kind, thanks to the great potenziality of the electronic dispositives.

1. TIME FREQUENCY-DIAGRAMS.

An intuitive way to visualize a translation or a reflection of the usual euclidean plane is offered by the following diagrams (see Choquet (1964), Weyl (1952)).

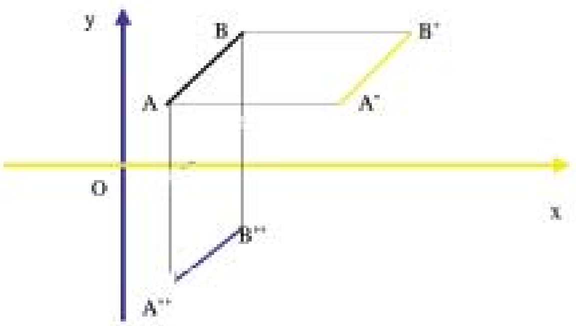

Figure 1.1. *Translation.* A'B' is translated by AB long the x axis , in the creasing direction. A''B'' is translated by AB long the y axis, in the decreasing direction.

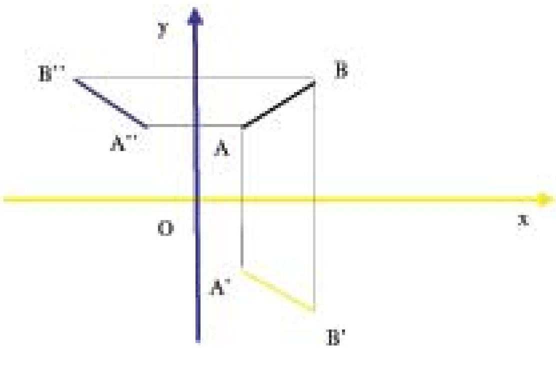

Figure 1.2. *Reflection.* A'B' is reflected by AB respect to the x axis. A''B'' is reflected by AB respect to the y axis.

It is possible to discover a logic of this kind in many pieces of J.S.Bach and in the entire Musical Offering.

Introducing a monometric orthogonal reference frame in the plane, we can visualize the temporal evolution of an hand which plays the ascending scale of C major in the central octave of the piano. Fixed a unity of time, for istance the quaver, we grade the x axis (Duration Axis) with integer multiple of the fixed unity. On the y axis (High Axis) we grade according to the frequencies of the white tastes of the central octave of the piano. Roughly speaking, each sound is characterized by hight, intensity and timbre, in particular the hight is an acoustic size which is measured in Hertz (see Frova (1990), Mazzola (2002), Russo (2004)).

The range which competes to the white tastes of the central octave of the piano goes from 264 to 520 Hertz in the sequence do=C=264Hz, re=D=297Hz, mi=E=330Hz, fa=F=352Hz, sol=G=396Hz, la=A=440Hz, si=B=495Hz, do=C=520Hz.

We remark that the black tastes and other frequencies between two consecutive white tastes of the central octave of the piano are not showed only to semplify the discourse.

Fig.1.3 describes a common exercise of piano: the ascending scale of C major, played by a single hand in the central octave. The usual succesive exercise, the descending and ascending scale of C major played by a single hand in the central octave, is described by the diagram in Fig.1.4. **Fig.1.2**

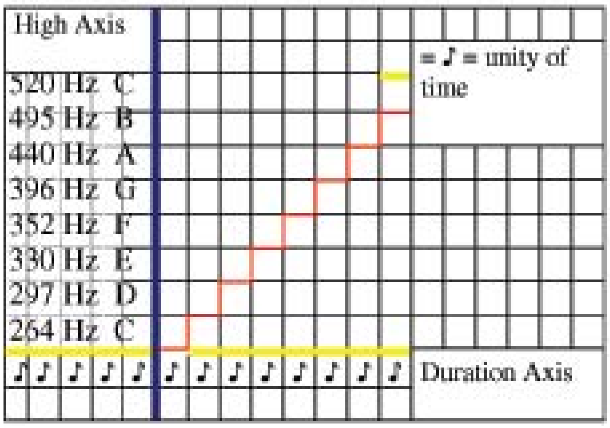

Figure.1.3. *Graphic representation in a time-frequency diagram of the ascending scale of C major.*

In this monometric orthogonal reference frame is used the quaver as unity of time for the Duration Axis, while on the High Axis there is a band of frequencies, which is associated with the white tastes of the central octave of the piano.

In Fig.1.4 the blue line is reflected by the red line long the High Axis as in Fig.1.2, so a common exercise of piano is reduced to a reflection in mathematical language. In Figg.1.4 and 1.5 there is only one instrument which plays (left hand on a piano). The natural question is to see what happens when 2 or more instruments (voices) are playing, describing translations and reflections in their melody. This is the intuitive idea of a Canon.
A detailed description of these examples of simmetries in Music can be found in Galante (1998), Mazzola (2002), Russo (2004), Scimemi (1983, 1992,1997).

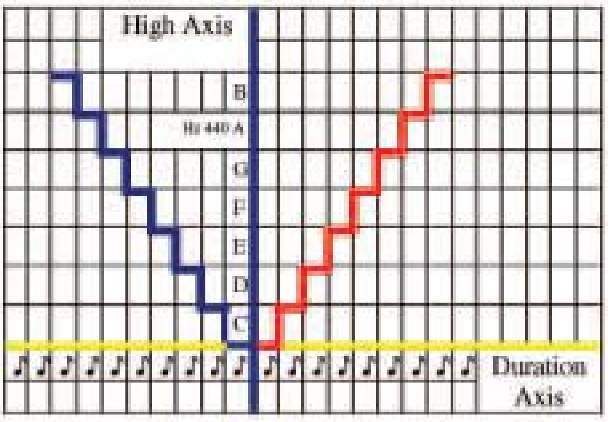

Figure 1.4. *Graphic representation in a time-frequency diagram of the descending and ascending scale of C major.*

2. CANONS OF THE MUSICAL OFFERING, AFFINITIES AND ISOMETRIES.

Canons and other musical forms of the Musical Offering are composed with a special tecnique of the Harmony called *imitation*: there are two or more voices (instruments) which play, making interweavings, translations (moto parallelo) and reflections (moto contrario). We refer to Basso (1985), Karolyi (1969), Schweitzer (1967) for a specific definition of Canon.
Criteria of imitation, via time-frequency diagrams, have a mathematical expression with affinities and isometries of the euclidean plane Π (see Choquet (1964), Robinson (1980), Weyl (1954)). This correspondence allows to write the equation of government of an arbitrary Canon with an arbitrary number of voices. Hence the classification of a Canon is reduced to the classification of its associated transformation in Π.
Here we see how it is possible to write the equation of government with an algorithm.

Step1. We fix a suitable reference frame with ordinates the High Axis (equable temperament) and with abscisses the Duration Axis (the smallest value). This step means that we will use only time frequency-diagrams.

Step2. We fix a voice, i.e.1[st] -voice, and we choose on it the first non collinear points $(d_{1,1}, h_{1,1})$, $(d_{1,2}, h_{1,2})$, $(d_{1,3}, h_{1,3})$ of Π.
To determine the affinity of Π which sends the 1[st]-voice to the n[th]-voice of the canon, we choose the first non collinear points $(d_{n,1}, h_{n,1})$, $(d_{n,2}, h_{n,2})$, $(d_{n,3}, h_{n,3})$ of the n[th]-voice and we impose to the arbitrary affinity

$$\begin{cases} d' = d_0 + \alpha d + \beta h \\ h' = h_0 + \gamma d + \delta h \end{cases} \qquad \textbf{(2.1)}$$

with $(\alpha\delta - \beta\gamma \neq 0)$ to send $(d_{1,i}, h_{1,i})$ in $(d_{n,i}, h_{n,i})$ where $i = 1,2,3$.

Step3. We obtain the linear system of 6 equations in 6 unkown $d_0, h_0, \alpha, \beta, \gamma, \delta$:

$$\begin{cases} d_0 + d_{1,1}\alpha + h_{1,1}\beta = d_{n,1} \\ d_0 + d_{1,2}\alpha + h_{1,2}\beta = d_{n,2} \\ d_0 + d_{1,3}\alpha + h_{1,3}\beta = d_{n,3} \\ h_0 + d_{1,1}\gamma + h_{1,1}\delta = h_{n,1} \\ h_0 + d_{1,2}\gamma + h_{1,2}\delta = h_{n,2} \\ h_0 + d_{1,3}\gamma + h_{1,3}\delta = h_{n,3} \end{cases} \qquad \textbf{(2.2)}$$

Since $(d_{1,1}, h_{1,1})$, $(d_{1,2}, h_{1,2})$, $(d_{1,3}, h_{1,3})$ are not collinear, the coefficient matrix (associated with the homogeneous linear system)

$$\begin{pmatrix} 1 & d_{1,1} & h_{1,1} & 0 & 0 & 0 \\ 1 & d_{1,2} & h_{1,2} & 0 & 0 & 0 \\ 1 & d_{1,3} & h_{1,3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & d_{1,1} & h_{1,1} \\ 0 & 0 & 0 & 1 & d_{1,2} & h_{1,2} \\ 0 & 0 & 0 & 1 & d_{1,3} & h_{1,3} \end{pmatrix} \qquad \textbf{(2.3)}$$

has rank 6;
Hence **(2.2)** has a unique solution

$$(d_{0,n}, \alpha_n, \beta_n, h_{0,n}, \gamma_n, \delta_n)$$

which can be found with the Cramer rule, where n is a fixed positive integer.

Step4. The n[th]-voice of the canon is obtained by the 1[th]-voice via the affinity

$$\begin{cases} d' = d_{0,n} + \alpha_n d + \beta_n h \\ h' = h_{0,n} + \gamma_n d + \delta_n h \end{cases} \qquad \textbf{(2.4)}$$

(*Equation of government of the* n[th]*-voice of the Canon*)

3. STRUCTURAL THEOREM OF ISOMETRIES OF Π.

An isometry of the euclidean plane Π is a rigid transformation, that is, a law which allows to send a set of points of Π in another set of points of Π without deformations along the whole process. A famous theorem of Plane Geometry states that all the isometries of Π coincide with suitable products of only reflections (see Choquet (1964) or Weyl (1952)). A statement of this result is the following.

Theorem 3.1. *Each isometry of Π in Π either is an identity or a product of the form s1, s1 o s2, s1 o s2 o s3, where si are reflections for i = 1,2,3. For all subset X ⊂ Π, each isometry f of X in Π can be extended to an isometriy g of Π in Π in a unique way, whenever X is not collinear, in two ways, whenever X is collinear and it contains at least two points.*

The musical interpretation of Theorem 3.1 is very interesting. Many harmonists and musicians of the Baroque period composed canons and similar pieces giving an enigma (see Basso (1985), Hofstadter (2001), Scwheitzer (1967)) or inserting some expressions before the Canon ("per contrarium moto", "in unisono" and so on). These expressions are the key to solve the Canon, that is, the way to explicit the melody of the voices when 1st-voice is given. In mathematical terms an enigma, which precedes a Canon, can be translated in reflections or product of them. They are uniquely associated with the Canon, according to Section 4, then a classification of a Canon based on its associated isometry via **(2.4)** gives the usual harmonic classification and conversely (see also Russo (2004)).

4. ALGEBRAIC-GEOMETRIC ANALYSYS OF THE CANONS OF THE MUSICAL OFFERING OF J.S.BACH.

Here we sketch the algebraic-geometric analysis of the Canon n.2 with 2 violins of the Musical Offering of J.S.Bach. The choice of this Canon is due to its easy structure and we apply the process described in Section 2. The generality of the process allows to repeat mutatis mutandis the same argument to each Canon of the Musical Offering.

When we listen the Musical Offering, it is necessary to note that there is one motive (*thema regium*) which often is changed and deformed by J.S.Bach. The tradition wants that thema regium was played by king Friederich, the Great of Prussia, during a visit of Bach at his court (see Schweitzer (1967)).

Canon n. 2 with 2 violins of the Musical Offering of J.S.Bach.

Figure.4.1. *Philological score of the Canon n.2 with 2 violins of the Musical Offering of J.S.Bach.*

The first two rows on the top (1st - voice and 2nd - voice) correspond in order to the part, which is played by violin I and by violin II. The down row corresponds to Basso Continuo, whose realization is given by piano.

The imitation which happens between the two voices (violin I and violin II) of the Canon is governed by the following equation, which describes a translation of Π :

$$\tau \quad \begin{cases} d_2 = d_1 + 16 \\ h_2 = h_1 \end{cases} \qquad \textbf{(4.1)}$$

where the unknown d_2 and h_2 show duration and height of the 2nd-voice in Fig. 4.1 and the values d_1 and h_1 are fixed and they show the duration and height of the 1st-voice. Equation **(4.1)** means a simple fact which we listen: the violin II starts after 16 semiquaver pauses at the same height of the violin I.

The translation vector of **(4.1)** is (16, 0) and it is parall with Duration Axis in the reference frame which we have previously described.

According to the proceeding to construct a melodically affine Canon in Section 2, Equation **(2.5)** was born choosing the first three non collinear points in Fig. 4.2 on 1st-voice (blue line), i.e (4,10), (8, 3), (14, 5).

We impose to **(2.1)** to send them in the first three non collinear points on 2nd -voice (green line), i.e. (20,10), (24, 3), (30, 5). Then we obtain the linear system **(2.2)** of 6 equations in 6 unknowns which has the unique solution given by the values

$$\alpha_2 = 1, \beta_2 = 0, \gamma_2 = 0, \delta_2 = 1, d_{0,2} = 16, h_{0,2} = 0.$$

In this way **(2.1)** assumes the form of **(2.4)** where the unknown d e h are respectively recalled d_1 e h_1 and d'= d_2 e h'= h_2. This leads to **(4.1)**.

Classification of the Canon n. 2 with 2 violins of the Musical Offering of J.S.Bach

(4.1) is a translation τ which is parallel with Duration Axis in the time-frequency diagram previously described. The structural theorem for isometries of Π

(Theorem 3.1) allows to say that τ can be reduced at the product of two successive reflections which have parallel axis whose direction is ortoghonal with Duration Axis. This description is characteristic for τ hence for the Canon.

Canon n. 2 with 2 violins of the Musical Offering of J.S.Bach.

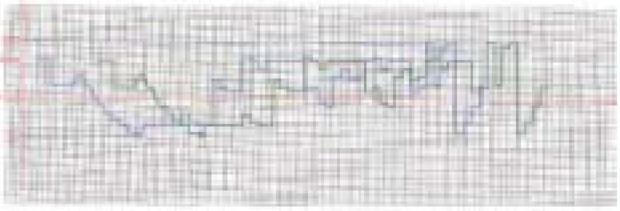

Figure.4.2. *Graphic representation in a time – frequency diagram of violin I and violin II in the Canon n. 2 with 2 violins of the Musical Offering of J.S.Bach..*

1) The unity of time for Duration Axis is the least value of time which is present in the score, that is, the semiquaver. Each other note has duration which is multiple of semiquaver.
2) On High Axis we have adopted a frequency's band which covers the extension of 2 octaves on the piano between 264 and 1046 Hz. This band describes black and white tastes (semipitches of equable temperament).
3) Origin of the reference frame is A at 440 Hz.
4) The values which we have marked on High Axis correspond at the attac instant and at the instants, which are related with the highest and lowest pitches in the execution of the piece.

5.CONCLUSIONS.

The method described to analize the Canon n. 2 with 2 violins of the Musical Offering of J.S.Bach in Fig. 4.1 can be extended to Canons at an arbitrary number of voices (*n-voices Canon*, n positive integer) where it is adopted an imitation tecnique. As before we fix the 1st -voice, then we write (n-1)-equations of the kind **(2.5)** in their general form:

$$\begin{cases} d_2 = a\,d_1 + b \\ h_2 = c\,h_1 + d \end{cases} \qquad \textbf{(5.1)}$$

being a , b, c, d real numbers and where the unknown d_2 and h_2 show the duration and height of the 2nd -voice. The values d_1 and h_1 are fixed and they show duration and height of the 1st -voice. The recognition of isometries and affinities in Equation **(5.1)**, thanks to Theorem 3.1, means to harmonically classify the Canon.

On the other hand it is interesting to note that a similar approach to the musical form Canon allows a method to compose. The assignment of the coefficients a , b, c, d and the values d_1 and h_1 in **(5.1)** means to construct canons with two voices, resolving the associated linear system. It is clear what is the argument, when we want to write the government equation of a n-voices canon.

A treatment with linear system, which is described here, can be simulated by elementary algorithms (Gauss Algorithm); Assayag & others (2002), Benson (2003), Mazzola (2002) give more informations about this computational aspect.

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