A Note on the Exterior Centralizer

Peyman Niroomand and Francesco Russo

Abstract. The notion of exterior centralizer $C^{\wedge}_G(x)$ of an element x of a group G is introduced in the present paper in order to improve some known results on the non-abelian tensor product of two groups. We study the structure of G by looking at that of $C^{\wedge}_G(x)$ and we find some bounds for the Schur multiplier $M(G)$ of G.

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1. Introduction

The non-abelian tensor square $G \otimes G$ of the group G is a group generated by the symbols $g \otimes h$ and subject to the relations

$$gg' \otimes h = ({}^{g}g' \otimes {}^{g}h)(g \otimes h) \quad , \quad g \otimes hh' = (g \otimes h)\ ({}^{h}g \otimes {}^{h}h') \tag{1.1}$$

for all $g, g', h, h' \in G$, where G acts on itself by conjugation via ${}^{g}g' = gg'g^{-1}$. The tensor square is a special case of the non-abelian tensor product $G \otimes H$ of the groups G and H. This notion was introduced in [5] and some significant results can be found in [3, 4, 6]. A classical survey on the topic is [7]. The exterior square $G \wedge G$ is obtained with the additional relation $g \otimes g = 1_{\otimes}$ on $G \otimes G$.

Recall that [3] describes the maps $\kappa : g \otimes h \in G \otimes G \longmapsto [g, h] = g^{-1}h^{-1}gh \in G'$ and $\kappa' : g \wedge h \in G \wedge G \longmapsto [g, h] \in G'$, which are both homomorphisms of groups. We write $\ker \kappa = J_2(G)$; its topological interest is in the formula $\pi_3 SK(G, 1) = J_2(G)$ (see [4]), which allows us to consider non-abelian tensor products as a powerful instrument both in Topology and in the Theory of Groups. Results in [3] give the following commutative diagram with exact rows and central extensions as columns:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
H_3(G) & \longrightarrow & \Gamma(G/G') & \xrightarrow{\psi} & J_2(G) & \longrightarrow & H_2(G) & \longrightarrow & 0 \\
\| & & \| & & \downarrow & & \downarrow & & \\
H_3(G) & \longrightarrow & \Gamma(G/G') & \xrightarrow{\psi} & G \otimes G & \longrightarrow & G \wedge G & \longrightarrow & 1 \\
 & & & & {\scriptstyle\kappa}\downarrow & & {\scriptstyle\kappa'}\downarrow & & \\
 & & & & G' & = & G' & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array} \tag{1.2}$$

where $H_n(G,\mathbb{Z})$ is the nth integral homology group of G and Γ is the Whitehead's quadratic functor in [9]. In this context, if x is an element of G, we may consider the *tensor centralizer*

$$C_G^{\otimes}(x)=\{a\in G|a\otimes x=1_{\otimes}\} \quad (1.3)$$

of x in G and the *tensor center*

$$Z^{\otimes}(G)=\{g\in G|1_{\otimes}=g\otimes y\in G\otimes G,\forall y\in G\} \quad (1.4)$$

of G. We may already find these notions in [2] and [6, Section 4]. [2, Corollary 3.3] shows $Z^{\otimes}(G)=\bigcap_{x\in G}C_G^{\otimes}(x)$. Assigned an element g of G, the tensor centralizer and the tensor center are related to the subgroup

$${}^{\otimes}E(G,g)=\{a\in G|xa\otimes g=x\otimes g,\forall x\in G\} \quad (1.5)$$

of G and to the subgroup

$$E^{\otimes}(G,g)=\{a\in G|ax\otimes g=x\otimes g,\forall x\in G\} \quad (1.6)$$

of G by the formula $Z^{\otimes}(G)=\bigcap_{g\in G}{}^{\otimes}E(G,g)=\bigcap_{g\in G}E^{\otimes}(G,g)$ of [2, Corollary 4.4].

The operator $\wedge$ was already investigated in [6] in this context. More precisely, we may consider the *exterior centralizer*

$$C_G^{\wedge}(x)=\{a\in G|a\wedge x=1_{\wedge}\} \quad (1.7)$$

of x in G and the *exterior center*

$$Z^{\wedge}(G)=\{g\in G|1_{\wedge}=g\wedge y\in G\wedge G,\forall y\in G\} \quad (1.8)$$

of G. We will see that $Z^{\wedge}(G)=\bigcap_{x\in G}C_G^{\wedge}(x)$. Already [6, Proposition 16 (i)] shows that $Z^{\wedge}(G)$ is a subgroup of $Z(G)$ which allows us to decide whether G is a capable group, that is, whether G is isomorphic to $E/Z(E)$ for some group E. See [6, Proposition 16] or [1].

In analogy with [2, Section 4], given an element g of G, we may consider

$$E^{\wedge}(G,g)=\{a\in G:xa\wedge g=x\wedge g,\forall x\in G\} \quad (1.9)$$

and check easily that $E^{\wedge}(G,g)$ is a subgroup of G.

To the best of our knowledge, we have noted that there is not a wide literature on (1.3), involving the operator $\wedge$ instead of $\otimes$. The same happens for (1.5) and (1.6). Here we formalize such concepts and we find some bounds for the Schur multiplier $M(G)$ of G. Our terminology is standard and follows [3, 4, 7, 8, 9].

2. Some results of general interest

It is easy to check the following fact.

Lemma 2.1. *$C_G^{\wedge}(x)$ is a subgroup of a group G. Furthermore, $C_G^{\wedge}(x)$ is normal in $C_G(x)$.*

Proof. Let $a\in C_G^{\wedge}(x)$. We have $\kappa'(a\wedge x)=[a,x]=1_G$. Hence $C_G^{\wedge}(x)$ is contained in $C_G(x)$. On the other hand, $1_G\wedge x=1_{\wedge}\in C_G^{\wedge}(x)$ so that $C_G^{\wedge}(x)$ is a non-empty set. Now, let $a,b\in C_G^{\wedge}(x)$. We have

$$ab^{-1}\wedge x={}^{a}(b^{-1}\wedge x)(a\wedge x)={}^{ab^{-1}}(b\wedge x)^{-1}\quad(a\wedge x)=1_{\wedge}$$

so $ab^{-1}\in C_G^{\wedge}(x)$ and we may conclude that $C_G^{\wedge}(x)$ is a subgroup of G.

If $g\in C_G^{\wedge}(x)$ and $y\in C_G(x)$, then

$${}^{y}g\wedge x={}^{y}(g\wedge x)=1_{\wedge}$$

thus $C_G^{\wedge}(x)$ is normal in $C_G(x)$. □

The next lemma modifies [2, Corollary 4.4].

Lemma 2.2. *$E^{\wedge}(G,g) = C^{\wedge}_{G}(g)$ and $Z^{\wedge}(G) = \bigcap_{g\in G} E^{\wedge}(G,g)$.*

Proof. First of all, $1_G 1_G \wedge g = 1_G \wedge g = 1_{\wedge} \in E^{\wedge}(G,g)$ so that $E^{\wedge}(G,g)$ is a non-empty subset of G. Now, we claim that $E^{\wedge}(G,g) = C^{\wedge}_{G}(g)$. Since $xa \wedge g = ^{x}(a \wedge g)(x \wedge g)$ for x and a in G, the definitions of $E^{\wedge}(G,g)$ and $C^{\wedge}_{G}(g)$ give that $a \in E^{\wedge}(G,g)$ if and only if $a \in C^{\wedge}_{G}(g)$. The last part of the lemma follows obviously. □

Lemma 2.2 gives as follows.

Lemma 2.3. *Let N be a normal subgroup of G. Then the following sequences are exact*

(i) $1 \longrightarrow C^{\wedge}_{G}(x) \cap N \longrightarrow C^{\wedge}_{G}(x) \longrightarrow C^{\wedge}_{G/N}(xN)$,

(ii) $1 \longrightarrow Z^{\wedge}(G) \cap N \longrightarrow Z^{\wedge}(G) \longrightarrow Z^{\wedge}(G/N)$.

Proof. (i). Consider the natural epimorphism $\pi : g \wedge h \in G \wedge G \mapsto gN \wedge hN \in G/N \wedge G/N$. If $y \in C^{\wedge}_{G}(x)$, then $\pi(y) = yN \in C^{\wedge}_{G/N}(xN)$. On the other hand, if $\pi(y) = 1$ for $y \in C^{\wedge}_{G}(x)$, then $y \in N \cap C^{\wedge}_{G}(x)$. The result follows.

(ii). It follows from (i) above and Lemma 2.2. □

We can form quotients of (1.7) as follows.

Remark 2.4. Let N be a normal subgroup of G. From Lemma 2.3, $N \leq C^{\wedge}_{G}(x)$ if and only if $C^{\wedge}_{G}(x)/N = C^{\wedge}_{G/N}(xN)$. In particular, we have that $N \leq C^{\wedge}_{G}(x)$ for all $x \in G$ if and only if $Z^{\wedge}(G)/N = Z^{\wedge}(G/N)$.

The following corollary adapts [1, Corollary 2.2] to (1.8).

Corollary 2.5. *$Z^{\wedge}(G)$ is the smallest subgroup of G such that $G/Z^{\wedge}(G)$ is a capable group.*

Proof. Let N be a normal subgroup of G and G/N be a capable group. Then $Z^{\wedge}(G/N)$ is trivial by [6, Proposition 16 (vii)] and $Z^{\wedge}(G)N/N \leq Z^{\wedge}(G/N)$. Therefore, $Z^{\wedge}(G) \leq N$. On the other hand, $G/Z^{\wedge}(G)$ is a capable group because we have that $Z^{\wedge}(G/Z^{\wedge}(G)) = Z^{\wedge}(G)/Z^{\wedge}(G)$ is trivial. □

We may split the exterior centralizers as follows.

Proposition 2.6. *Assume that G and H are finite groups such that $(|H|,|G|) = 1$. If $x \in G$ and $y \in H$, then $C^{\wedge}_{G\times H}(xy) = C^{\wedge}_{G}(x) \times C^{\wedge}_{H}(y)$ and $Z^{\wedge}(G \times H) = Z^{\wedge}(G) \times Z^{\wedge}(H)$*

Proof. It is enough to prove that $C^{\wedge}_{G\times H}(xy) = C^{\wedge}_{G}(x) \times C^{\wedge}_{H}(y)$. Let $a \in C^{\wedge}_{G}(x)$ and $b \in C^{\wedge}_{H}(y)$. Of course, $a \wedge xy = 1_{\wedge}$ implies both $a \wedge x = 1_{\wedge}$ and $a \wedge y = 1_{\wedge}$. Similarly, this happens for b. Then it is clear that $C^{\wedge}_{G\times H}(xy) \leq C^{\wedge}_{G}(x) \times C^{\wedge}_{H}(y)$.

Conversely, if $a \in C_G(x)$ and $b \in C_H(y)$, then

$$ab \wedge xy = [b \wedge x][b \wedge y][a \wedge x][a \wedge y] = 1_{\wedge},$$

since $(|\langle a\rangle|, |\langle b\rangle|) = 1$.

Thus $ab \in C^{\wedge}_{G\times H}(xy)$ and so $C^{\wedge}_{G}(x) \times C^{\wedge}_{H}(y) \leq C^{\wedge}_{G\times H}(xy)$. □

Now, we may give a structural result which is related to $M(G)$.

Proposition 2.7. *$C_G(x)/C^{\wedge}_{G}(x)$ is isomorphic to a subgroup of $M(G)$.*

Proof. We may easily check that the map

$$f : y \in C_G(x) \longmapsto f(y) = x \wedge y \in M(G)$$

is a homomorphism of groups. Note that $M(G)$ is a factor group of $J_2(G)$ (see (1.2) or directly [3]) and we know from [4] that $J_2(G)$ is a central subgroup of $G \otimes G$ and that the elements of $J_2(G)$ are fixed under the action of G. This allows us to conclude that the elements of $M(G)$ are fixed by the action of G. On the other hand, it is clear that the kernel of f is $C^{\wedge}_{G}(x)$. Then the result follows. □

Proposition 2.7 has an interesting consequence as follows.

Corollary 2.8. *If G has an element x such that $C_G(x) \neq C_G^{\wedge}(x)$, then $M(G)$ is non-trivial.*

[6, Proposition 16] shows that a finite group G has non-trivial $M(G)$, whenever $Z^{\wedge}(G) \neq Z(G)$. The group $G = (C_2 \times Q_8) \rtimes C_2$ has order 32, $Z^{\wedge}(G) = Z(G) = C_2$ and non-trivial $M(G)$, because there exists an element x such that $C_G(x) \neq C_G^{\wedge}(x)$. Then we may conclude that Corollary 2.8 strengthens [6, Proposition 16].

Define the set

$$K = \bigcap_{x \in Z(G)} C_G^{\wedge}(x) \tag{2.1}$$

It is easy to check that K is a normal subgroup of G. Of course, if G is an abelian group, then $K = Z^{\wedge}(G)$. A useful property of K is the following.

Lemma 2.9. *Consider K in (2.1). Then $G' \leq K$.*

Proof. Let x be an element of $Z(G)$ and y, g be elements of G. We have

$$[y, g] \wedge x = {}^{x}(y \wedge g)(y \wedge g)^{-1}.$$

Since $x \in Z(G)$, we have $[y, g] \wedge x = 1_{\wedge}$. More generally, we have

$$[x_1, y_1]^{\alpha_1}[x_2, y_2]^{\alpha_2} \dots [x_n, y_n]^{\alpha_n} \wedge x = 1_{\wedge},$$

where $x_1, y_1, x_2, y_2, \dots, x_n, y_n$ are elements of G, $n \geq 1$ is an integer and $\alpha_1, \alpha_2, \dots, \alpha_n$ are integers. From this, we deduce that $G' \leq C_G^{\wedge}(x)$ for every element x of $Z(G)$. □

3. Restrictions on the Schur multiplier

This Section illustrates our main results. We will see that the considerations in Section 2 allow us to control the size of $M(G)$ and to decide when a group is capable.

Proposition 3.1. *Let G be a group such that $C_G(x) = C_G^{\wedge}(x)$, for all x not belonging to $Z(G)$. Then $G' \cap Z(G)$ is a subgroup of $Z^{\wedge}(G)$. In particular, if G is a capable group, then $G' \cap Z(G)$ is trivial.*

Proof. Assume that $C_G(x) = C_G^{\wedge}(x)$, where $x \notin Z(G)$. Then $Z(G) \leq \bigcap_{x \notin Z(G)} C_G^{\wedge}(x)$. On the other hand, Lemma 2.9 implies that $G' \leq \bigcap_{x \in Z(G)} C_G^{\wedge}(x)$. Therefore,

$$Z(G) \cap G' \leq \bigcap_{x \in G} C_G^{\wedge}(x) = Z^{\wedge}(G),$$

as claimed. The remaining part of the result is obvious. □

An easy consequence of Proposition 3.1 is the following.

Corollary 3.2. *If G is a nilpotent capable group, then there is a non-central element x of G such that $C_G(x) \neq C_G^{\wedge}(x)$.*

The subgroup K in (2.1) allows us to have structural information on G, once we know some relations with respect to $Z^{\wedge}(G)$. More precisely, the following two statements inform us when a group G is abelian.

Corollary 3.3. *Let K be the subgroup in (2.1). If $Z^{\wedge}(G) = K$, then G is nilpotent of class at most 2. Furthermore, if $M(G) = M(G/K)$, then G is an abelian group.*

Proof. Since $K = Z^{\wedge}(G)$, G' is a central subgroup of G. Therefore, G is a nilpotent group of class at most 2. On the other hand, we know that the Ganea sequence is exact. See [8, Theorem 2.5.6 (i)]. Since $M(G) = M(G/K)$, G' is trivial. The result follows. □

Recall that a group G is called *unicentral* if $Z^{\wedge}(G) = Z(G)$. The structure of unicentral groups is described in [1] and [6, Section 4]. Then we have the next result, whose proof is straightforward.

Corollary 3.4. *Let K be the subgroup in (2.1). $K = G$ if and only if G is a unicentral group.*

We recall that $d(G)$ is the minimum number of generators of a group G.

Theorem 3.5. *Assume that G is a group with finite $M(G)$ and with finite $d(G/Z^{\wedge}(G))$. Then $|Z(G)/Z^{\wedge}(G)|$ divides $|M(G)|^{d(G/Z^{\wedge}(G))}$.*

Proof. Assume that $G/Z^{\wedge}(G) = \langle \bar{x}_1, \ldots \bar{x}_n\rangle$ for some integer $n \geq 1$ and elements $\bar{x}_1 = x_1Z^{\wedge}(G), \ldots, \bar{x}_n = x_nZ^{\wedge}(G)$ of $G/Z^{\wedge}(G)$. Define

$$\varphi : x \in Z(G) \longmapsto (x \wedge x_1, \ldots, x \wedge x_n) \in M(G)^n.$$

We deduce that φ is a homomorphism of groups, because $x, y \in Z(G)$ imply that

$$xy \wedge x_i = (x \wedge x_i) \quad {}^x(y \wedge x_i) = (x \wedge x_i)(y \wedge x_i)$$

for every $i \in \{1, \ldots n\}$.

Now, we claim that $\ker\varphi = Z^{\wedge}(G)$. It is easy to check that $Z^{\wedge}(G) \leq \ker\varphi$. On the other hand, if $x \in \ker\varphi$, then $x \wedge x_i = 1$ for every $i \in \{1, \ldots n\}$. It is enough to show that $x \wedge y = 1$ for every $y \in G$ in order to finish our proof. If $y \in G\setminus Z^{\wedge}(G)$, then the assumptions imply that $y = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$, where $\alpha_1, \ldots \alpha_n$ are integers. Thus,

$$x \wedge y = x \wedge (x_1^{\alpha_1} \ldots x_n^{\alpha_n}) = (x \wedge x_1^{\alpha_1}) \ldots (x \wedge x_n^{\alpha_n}) = (x \wedge x_1)^{\alpha_1} \ldots (x \wedge x_n)^{\alpha_n} = 1_{\wedge}.$$

The result follows. □

Note that Theorem 3.5 holds in the case of finite groups and in the case of polycyclic groups. We have an interesting consequence for groups having trivial Schur multiplier.

Corollary 3.6. *A group G in the hypothesis of Theorem 3.5 with trivial Schur multiplier has $C_G(x) = C^{\wedge}_G(x)$ for every element x of G. In particular, such a group has $Z(G) = Z^{\wedge}(G)$.*

Example. The group $G = C_2 \times C_4$ has $Z^{\wedge}(G) = C_2$, $Z(G) = G$, $d(G/Z^{\wedge}(G)) = 2$, $M(G) = C_2$ and $Z(G)/Z^{\wedge}(G) \simeq C_4$. This group clarifies the situation in Theorem 3.5. For the non-abelian case, we can consider the semidirect product G of C_8 by C_4. We have that G is a group of order 32 with $Z(G) = C_2 \times C_4$, $Z^{\wedge}(G) = C_2$ and $d(G/Z^{\wedge}(G)) = 2$. In this case the equality holds.

We may improve Theorem 3.5 in case of finite groups as follows.

Corollary 3.7. *If G is a finite group with $G' \leq Z^{\wedge}(G)$, then $|G|$ divides*

$$(|M(G)||G'|)^{d(G/Z^{\wedge}(G))}|Z^{\wedge}(G)|.$$

If G is a capable group in Theorem 3.5, then $|Z(G)|$ divides $|M(G)|^{d(G/Z^{\wedge}(G))}$. This remark allows us a further improvement.

Proposition 3.8. *Let K be the subgroup in (2.1). If G is a capable group and satisfies the hypothesis of Theorem 3.5, then $|Z(G)|$ divides $|M(G)|^{d(G/K)}$.*

Proof. As in the proof of Theorem 3.5, we consider the homomorphism of groups

$$\varphi : x \in Z(G) \longmapsto (x \wedge y_1, \ldots, x \wedge y_t) \in M(G)^{d(G/K)},$$

where $t \geq 1$ is an integer and $\bar{y}_1 = y_1K, \ldots, \bar{y}_t = y_tK$ are the generators of G/K.

Now, if $x \in \ker\varphi$, then $x \wedge y_i = 1$ for $i \in \{1, \ldots, t\}$ and we know that $x \wedge g = 1$ for all $g \in G \setminus K$. If $g \in K$, then $g \in \bigcap_{y \in Z(G)} C^{\wedge}_G(y)$ and so $g \wedge y = 1$ for all $y \in Z(G)$. Therefore, we may conclude that $x \wedge g = 1$ for all $g \in G$ and so $x \in Z^{\wedge}(G)$. Since $Z^{\wedge}(G)$ is trivial, x has to be trivial and so $\ker\varphi$ is trivial. This proves that φ is a monomorphism of groups and our result follows. □

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Peyman Niroomand
School of Mathematics and Computer Science
Damghan University of Basic Sciences, Damghan, Iran
e-mail: `p_niroomand@yahoo.com`

Francesco Russo
University of Naples Federico II, Naples, Italy
e-mail: `francesco.russo@dma.unina.it`