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ON A SECTION WHICH IS CORRELATED TO THE HYPERQUASICENTER OF A GROUP

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Abstract

N. P. Mukherjee describes the hyperquasicenter $Q^*(G)$ of a finite group G in 1970 and 1972. J. C. Beidleman and H. Heineken extend this notion to the infinite case in 2001. Here we prove that the section $G/C_G(Q^*(G))$ is supersolvable under the maximal condition on the normal subgroups.

1. Introduction

Two subgroups H and K of a group G are said to *permute* if $HK = KH$. It is easy to see that H and K permute if and only if the set HK is a subgroup of G. A subgroup of G is *permutable* (or *quasinormal*) if it permutes with every subgroup of G. An element x of G is called *quasicentral* (q.c.) *element* if $\langle x\rangle$ is a permutable subgroup of G. The quasicenter of G, denoted by $Q(G)$, is the subgroup generated by all q.c. elements of G. The *hyperquasicenter* of G, denoted by $Q^*(G)$, is the

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largest term of the chain of normal subgroups of G,

$$1 = Q_0(G) \leq Q_1(G) = Q(G) \leq \cdots \leq Q_\alpha(G) \leq Q_{\alpha+1}(G) \leq \cdots,$$

where α is an ordinal,

$$Q_{\alpha+1}(G)/Q_\alpha(G) = Q(G/Q_\alpha(G)),$$

λ is a limit ordinal and

$$Q_\lambda(G) = \bigcup_{\alpha<\lambda} Q_\alpha(G).$$

A normal subgroup N of a group G is said to be *hypercyclically embedded in G* if it contains a G-invariant series whose factors are cyclic. In [3] it has been proved that $Q^*(G) = \sigma(G)$, where $\sigma(G)$ denotes the largest hypercyclically embedded normal subgroup of G (see [3, Theorem 1]). [1, 2, 5] show that for finite groups, [3, Theorem 1] continues to be valid if weaker conditions of permutability are considered. In this direction of research, it is well known that if G is a finite group, then $G/C_G(Q^*(G))$ is supersolvable (see [4, Theorem 6.9]). A generalization of [4, Theorem 6.9] for infinite groups represents the main result of the present paper.

Main Theorem. *Let G be a group. If G satisfies the maximal condition on the normal subgroups, then $G/C_G(Q^*(G))$ is supersolvable.*

After proving Main Theorem in Section 2, Section 3 shows that Main Theorem cannot be weakened. Most of our notation follows [10].

2. Proof of Main Theorem

The situation which is described in Main Theorem was already clear in the finite case by [1, 2, 5, 7, 8]. This is recalled before of proving Main Theorem.

Let G be a finite group, p be a prime and P be a p-subgroup of G. Then we write

$$\Omega(P) = \begin{cases} \Omega_1(P), & \text{if } p > 2, \\ \Omega_2(P), & \text{if } p = 2, \end{cases}$$

where $\Omega_1(G)$ is the subgroup of P generated by its elements of order dividing p and $\Omega_2(G)$ is the subgroup of P generated by its elements of order dividing p^2.

Let G be an arbitrary group. Following Kegel [6], a subgroup of G is called *S-quasinormal* in G if it permutes with every Sylow subgroup of G. Agrawal [1] defined the generalized center $\mathrm{genz}(G)$ of G to be the subgroup of G generated by all elements g of G such that $\langle g \rangle$ is S-quasinormal in G. The *generalized hypercenter* $\mathrm{genz}_\infty(G)$ is the largest term of the chain

$$1 = \mathrm{genz}_0(G) \leq \mathrm{genz}_1(G) = \mathrm{genz}(G) \leq \mathrm{genz}_2(G) \leq \cdots,$$

where

$$\mathrm{genz}_{\alpha+1}(G)/\mathrm{genz}_\alpha(G) = \mathrm{genz}(G/\mathrm{genz}_\alpha(G)),$$

α is an ordinal, $\mathrm{genz}_\lambda(G) = \bigcup_{\alpha<\lambda} \mathrm{genz}_\alpha(G)$ and λ is a limit ordinal. By definitions, $Q^*(G) \leq \mathrm{genz}_\infty(G)$. Note that $Q^*(G)$ is defined as the $Q_\infty(G)$ in [1, 2, 5, 7, 8]. The following two results are, respectively, [2, p. 2240, l.18-20] and [2, Theorem 3.11].

Proposition 2.1 [J. B. Derr, W. E. Deskins and N. P. Mukherjee]**.** *Let G be a finite group. If K is a normal subgroup of G such that G/K is supersolvable and $\Omega(P) \leq Q^*(G)$, for all Sylow subgroups P of K, then G is supersolvable.*

Proposition 2.2 [M. Asaad and M. Ezzat Mohamed]**.** *Let G be a finite group and p be a prime. If P is a normal p-subgroup of G such that G/P is supersolvable and $\Omega(P) \leq \mathrm{genz}_\infty(G)$, then G is supersolvable.*

Corollary 2.3. *Let G be a finite group. If $\Omega(P) \leq Q^*(G)$ for all Sylow subgroups P of $C_G(Q^*(G))$, then G is supersolvable.*

Proof. This follows by [4, Theorem 6.9] and Proposition 2.1.

Corollary 2.4. *Let G be a finite group and p be a prime. If* $C_G(Q^*(G))$ *is a p-group and* $\Omega(C_G(Q^*(G))) \leq Q^*(G)$, *then G is supersolvable.*

Proof. From [4, Theorem 6.9], $C_G(Q^*(G))$ is a normal p-subgroup of G such that $G/C_G(Q^*(G))$ is supersolvable. Since $\Omega(C_G(Q^*(G))) \leq Q^*(G)$ $\leq \mathrm{genz}_\infty(G)$, Proposition 2.2 proves the result.

Proof of Main Theorem. Let p be a prime and N be a supersolvably embedded normal subgroup of G. If G satisfies the maximal condition on the normal subgroups, then each series which lies in N with G-invariant terms and cyclic factors must reach N after finitely many steps. This means that there exist a positive integer n and a series

$$1 = K_0 \lhd K_1 \lhd \cdots \lhd K_{n-1} \lhd K_n = N$$

with G-invariant terms and factors K_i/K_{i-1} which are either cyclic of order p or infinite cyclic for each $1 \leq i \leq n$. In particular we conclude that N is supersolvable and every chief factor of G which lies in N has either order p or is infinite cyclic.

A famous result of G. Zappa (see [9, Theorem 3.11]) allows us to ordering the factors K_i/K_{i-1}: firstly the factors of order $p \neq 2$ in ascending sequence with respect to p, then the infinite cyclic factors, finally the factors of order 2.

In particular each minimal G-invariant subgroup K of N has either order p (eventually $p = 2$) or is infinite cyclic.

Our result will be proved, once that $G/C_G(N)$ is supersolvable.

Case 1. Assume that K has order $p \neq 2$. Thanks to the minimality of K, we may suppose that $K = K_1$. We proceed by induction on n. If $n = 1$, then $|N| = p$ and $G/C_G(N)$ is obviously supersolvable. Let $n > 1$ and assume by induction that $(G/L)/C_{G/L}(N/L)$ is supersolvable, where N/L is a proper quotient of N.

Of course, $G/C_G(K)$ is supersolvable. Put $D/K = C_{G/K}(N/K)$, then $C_G(N)K/K = D/K$, $D \triangleleft G$ and by induction $(G/K)/(D/K) \simeq G/D$ is supersolvable. Note that N is a (supersolvable) hypercyclically embedded normal subgroup of G and K is a minimal G-invariant subgroup of N, then $[K, N] = 1$ (see [10, pp. 217-218]), and so $K \leq C_G(N)$. We have that

$$C_G(N) \geq C_G(K) \cap D = C_G(K) \cap C_G(N)K = C_G(K) \cap C_G(N).$$

It follows that

$$G/C_G(N) \simeq \frac{G/C_G(K) \cap D}{C_G(N)/C_G(K) \cap D}.$$

It is enough to prove that $G/C_G(K) \cap D$ is supersolvable. This is true because $G/C_G(K) \cap D$ is isomorphic to a subgroup of the direct product of the two supersolvable groups $G/C_G(K)$ and G/D. Therefore, $G/C_G(N)$ is supersolvable, as claimed.

Case 2. Assume that K is infinite cyclic and $K = K_1$. We proceed by induction on n applying the same argument of the previous Case 1.

Case 3. Assume that K has order 2 and $K = K_1$. We proceed by induction on n applying the same argument of the previous Case 1.

3. Examples

This section helps us to describe Main Theorem more concretely.

Example 3.1 [J. G. Thompson and K. Nakamura]**.** The present example has been introduced simultaneously by J. G. Thompson and K. Nakamura (see [10, Exercise 1, p. 227]). Let $p > 2$ be a prime and $A = \langle a_0, a_1, \ldots, a_p \rangle$ be an elementary abelian p-group of order p^{p+1}.

There exists an extension $H = A\langle x \rangle$ of A by a cyclic group $\langle x \rangle$ satisfying $x^{-1}a_ix = a_ia_{i-1}$, $x^{-1}a_0x = a_0$ and $x^{p^2} = a_0$ for each positive integer $i = 1, \ldots, p$. By construction H is hypercentral and has a series with H-invariant subgroups and cyclic factors. Then $Q^*(H) = H$. Here $C_H(Q^*(H)) = Z(H) = \langle a_0 \rangle$, $Z_2(H) = \langle a_0, a_1 \rangle$ and so on. H is hypercyclic and $H/C_H(Q^*(H))$ is supersolvable.

The infinite case of a group G, which is constructed following the Thompson-Nakamura method, does not allow us to state that the section $G/C_G(Q^*(G))$ is supersolvable. Let $p > 2$ be a prime and $B = \langle b_0, b_1, \ldots \rangle$ be an infinite elementary abelian p-group. There exists an extension $G = B\langle y \rangle$ of B by an infinite cyclic group $\langle y \rangle$ satisfying $y^{-1}b_j y = b_j b_{j-1}$, and $y^{-1}b_0 y = b_0$ for each positive integer $j = 1, 2, \ldots$. By construction G is hypercentral and has a series with G-invariant subgroups and cyclic factors. Then $Q^*(G) = G$. Here $C_G(Q^*(G)) = Z(G) = \langle b_0 \rangle$, $Z_2(G) = \langle b_0, b_1 \rangle$ and so on. Then G is hypercyclic, but $G/C_G(Q^*(G))$ is not supersolvable. We note that G does not satisfy the maximal condition on the normal subgroups, then Main Theorem cannot be applied.

It is useful to see that the infinite locally dihedral group D of type 2^∞ belongs to the groups described from Thompson-Nakamura. Also in this easy situation Main Theorem does not hold. Indeed, D does not satisfy the maximal condition on normal subgroups, $Q^*(D) = D$, $C_D(Q^*(D)) = Z(D)$ and $D/C_D(Q^*(D))$ is hypercyclic but not supersolvable.

Example 3.2. Given two distinct primes p and q, we consider the following class of finite groups:

$$\mathfrak{V}_1 = (a, b, c;\ a^p = b^p = c^q = 1;\ [a, b] = [b, c] = 1;$$

$$a^c = a^{-1};\ (p-1, q) = 1).$$

If $G \in \mathfrak{V}_1$, then $Q(G) = \langle a, b \rangle = G'$, $Z(G) = \langle b \rangle$, $Q^*(G) = G$ and G/Q is cyclic. Here $\langle a \rangle$ and $\langle b \rangle$ are permutable in G, but the product of $\langle a \rangle$ by $\langle ab \rangle$ is not a subgroup of G. $G/Z(G)$ is finite nonabelian of order pq. G is metabelian and nilpotent of length at most 3. We can consider for each positive integer i and j the following class of infinite groups, which generalizes $\mathfrak{V}_1$:

$$\hat{\mathfrak{V}}_1 = (a_i, b_j, c;\ a_i^p = b_i^p = c^q = 1;\ [a_i, b_j] = [b_j, c] = 1;$$

$$a_i^c = a_i^{-1};\ (p-1, q) = 1).$$

Also here for each group $G \in \hat{\mathfrak{V}}_1$ we have $Q(G) = \langle a_i, b_j : i, j \geq 1 \rangle = G'$, $Z(G) = \langle b_i : i \geq 1 \rangle$, $Q^*(G) = G$ and $G/Q(G)$ is cyclic. Here $\langle a_i \rangle$ and $\langle b_j \rangle$ are permutable in G for each i and j, but the product $\langle a_i \rangle \langle a_i b_j \rangle$ is not a subgroup of G for each i and j. $G/Z(G)$ is infinite nonabelian. G is metabelian and hypercyclic.

Example 3.3. The class of the extra-special groups (both finite and infinite) allows us to give a good description of $Z(G)$, $Q(G)$ and $Q^*(G)$. We recall that the symbol $\phi(G)$ denotes the Frattini subgroup of G. Consider $\mathfrak{V}_2 = (G' = Z(G) = \phi(G))$. For each group $G \in \mathfrak{V}_2$, $Q^*(G) = G$, either $Z(G) = G' = Q(G)$ or $Q(G) = G$, $G/Q(G)$ is abelian.

Example 3.4. The groups which act without fixed points on the additive group of a finite dimensional vector space have also a good description of $Z(G)$, $Q(G)$ and $Q^*(G)$. Let p, q and r be three distinct primes such that $(p, q, r) = 1$ and let

$$\mathfrak{V}_3 = (a, b, c, d;\ [b, c] = [b, d] = [c, d] = 1;\ a^p = b^q = c^r = d^r;$$

$$a^b = b^r;\ c^a = d;\ d^a = cd).$$

For each group $G \in \mathfrak{V}_3$ we have that $Q(G) = Q^*(G) = \langle b \rangle$, $Z(G) = 1$, $G/Q(G)$ is an elementary abelian r-group.

The infinite case of $\mathfrak{V}_3$ is given for each positive integer i and j by

$$\hat{\mathfrak{V}}_3 = (a, b, c_i, d_j; [b, c_i] = [b, d_j] = [c_i, d_j] = a^p = b^q = c_i^r = d_j^r = 1;$$

$$a^b = b^r; c_i^a = d_j; d_j^a = c_i d_j).$$

For each group $G \in \hat{\mathfrak{V}}_3$ we have that $Q(G) = Q^*(G) = \langle b_j : j \geq 1 \rangle$, $Z(G) = 1$, $G/Q(G)$ is an infinite elementary abelian r-group.

In the torsion-free case the situation is not so different. Let n be a positive integer and G be a group belonging either to the class $\hat{\mathfrak{V}}_4 = (\mathrm{Hol}(\mathbb{Q}))$ or to $\hat{\mathfrak{V}}_5 = (\mathrm{Hol}(\mathbb{Q}^n))$, where the symbol $\mathrm{Hol}(\mathbb{Q})$ denotes the holomorph of the additive group of the rational numbers and $\mathrm{Hol}(\mathbb{Q}^n)$ denotes the holomorph of the direct product of n-copies of the additive group of the rational numbers (see [10] for details). G has $Q(G) = Z(G) = Q^*(G) = 1$ and $G/Q(G)$ is not abelian.

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