Isoclinism in Probability of Commuting n-tuples

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Abstract

Strong restrictions on the structure of a group G can be given, once that it is known the probability that a randomly chosen pair of elements of a finite group G commutes. Introducing the notion of mutually commuting n-tuples for compact groups (not necessary finite), the present paper generalizes the probability that a randomly chosen pair of elements of G commutes. We shall state some results concerning this new concept of probability which has been recently treated in [3]. Furthermore a relation has been found between the notion of mutually commuting n-tuples and that of isoclinism between two arbitrary groups.

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1 Introduction

Let G be a finite group, then the probability that a randomly chosen pair of elements of G commutes is defined to be $\#com(G)/|G|^2$, where $\#com(G)$ is the number of pairs $(x, y) \in G \times G = G^2$ with $xy = yx$ and will be briefly denoted by $cp(G)$. From [6], one may easily find that $cp(G) = k(G)/|G|$, where $k(G)$ is the number of conjugacy classes of G. So, there is no ambiguity to use one or the other ratio in the universe of finite groups.

One way to generalize this probability is to consider n-tuples $(x_1, x_2, \ldots, x_n)$ of elements in a finite group G with the property that $x_ix_j = x_jx_i$ for all $1 \leq i, j \leq n$. Such n-tuples are called mutually commuting n-tuples. So, we may investigate the probability that randomly chosen ordered n-tuples of the group elements are mutually commuting n-tuples which we denote it by $cp_n(G)$. Note that for $n = 2$, this probability is exactly $cp(G)$.

For infinite groups, this ratio is not longer meaningful. In this case, compact groups with normalized Haar measure are good candidates for this procedure. As the similar description given in [3], we can define $cp_n(G)$. If G is a compact group with the normalized Haar measure μ, then it is possible to consider the product measure $\mu \times \mu$ on the product measure space $G \times G$ (see [8,Sections 18.1, 18.2] or [9, Chapter 2]). It is clear that $\mu \times \mu$ is again a probability measure. If

$$C_2 = \{(x, y) \in G \times G \mid xy = yx\},$$

then $C_2 = f^{-1}(1_G)$, where $f : G \times G \to G$ is defined via $f(x, y) = x^{-1}y^{-1}xy$ and 1_G denotes the neutral element of G. Obviously f is continuous and C_2 is a compact and measurable subset of $G \times G$. Therefore it is possible to define

$$cp(G) = (\mu \times \mu)(C_2).$$

Similarly, with the above notations, we may define $cp_n(G)$ in a compact group G, for all positive integers $n \geq 2$, as the following. If $\mu^n = \mu \times \mu \times \ldots \times \mu$ for n-times, then

$$cp_n(G) = \mu^n(C_n),$$

where

$$C_n = \{(x_1, \ldots, x_n) \in G^n \mid x_ix_j = x_jx_i \quad \text{for all} \quad 1 \leq i, j \leq n\}.$$

Obviously if G is finite, then G is a compact group with the discrete topology and so the Haar measure of G is the counting measure.

By the definitions it follows that for a compact group with a normalized Haar measure

$$cp_n(G) = \mu^n(C_n) = \frac{|C_n|}{|G|^n}$$

which is the same as in the finite case.

From the point of view of the compact groups, many results of [1,2,3,5,6,7,11,13] become special situations, since each finite group is trivially compact. In

1970 in [5], it has been proved that if G is a non-abelian finite group, then $cp(G) \leq 5/8$; furthermore this bound is achieved if and only if $G/Z(G)$ is isomorphic to an elementary abelian 2-group of rank 2, where $Z(G)$ denotes the center of the group G. Later, the first author and R. Kamyabi-Gol have extended this result in [3] to compact (not necessary finite, even uncountable) groups. For every non-abelian compact group G, they have proved that $G/Z(G)$ is isomorphic to an elementary abelian 2-group of rank 2 if and only if

$$cp_n(G) = \frac{3(2^{n-1}) - 1}{2^{2n-1}}$$

for all positive integers $n \geq 2$.

The present paper aims to improve the result of A.Erfanian and R. Kamyabi-Gol in two directions. First, we consider the case that $G/Z(G)$ is isomorphic to an elementary abelian p-group of rank 2, where p is a prime number and secondly, the case that $G/Z(G)$ is isomorphic to an elementary abelian p-group of rank k, where $k \geq 2$ is a positive integer. We will give the exact value of $cp_n(G)$ in both cases. Furthermore, we shall state a relation between the concept of isoclinism between groups (see [10]) and the above probability.

Our Main Theorems are :

Theorem A. *Let G be a non-abelian compact group (not necessary finite) and $G/Z(G)$ be a p-group, where p is a prime. Then the following statements are equivalent:*

(i) *$G/Z(G)$ is an elementary abelian p-group of rank 2;*

(ii) $cp_n(G) = \dfrac{p^n + p^{n-1} - 1}{p^{2n-1}}$, *where $n \geq 2$ is a positive integer;*

(iii) *G is isoclinic to an extra-special p-group of order p^3.*

Theorem B. *Let G be a non-abelian compact group, $r \geq 1$ be a positive integer and the index of $C_G(x)$ in G be a prime p for all $x \in G \backslash Z(G)$. Then the following statements are equivalent:*

(i) *$G/Z(G)$ is an elementary abelian p-group of rank $k = 2r$;*

(ii)

$$cp_n(G) = \frac{(p-1)\sum_{i=0}^{n-2} p^{i(k-1)} + p^{(n-1)k-n+2}}{p^{(n-1)k+1}},$$

where $n \geq 2$ is a positive integer;

(iii) *G is isoclinic to an extra-special p-group of order p^{k+1}.*

Section 2 gives preliminary results which are necessary to prove Main Theorems and Section 3 has been devoted to proof Main Theorems.

Most of our notation is standard and can be found in [9,13]. But, let us recall to define isoclinism between two groups for convenience of the reader:

a pair (φ, ψ) is called an *isoclinism* of groups G and H if φ is an isomorphism from $G/Z(G)$ to $H/Z(H)$, ψ is also an isomorphism from G' to H' and $\psi([g_1, g_2]) = [h_1, h_2]$ whenever $h_i \in \varphi(g_i Z(G))$, for all $g_i \in G$, $h_i \in H$, $i \in \{1, 2\}$. See [10] for details.

2 Preliminaries

In this Section, G is assumed to be a non-abelian compact group (not necessarily finite even uncountable) with normalized Haar measure μ. First, we state the following simple lemmas.

Lemma 2.1. *Let $C_G(x)$ be the centralizer of an element x in G. Then*

$$cp(G) = \int_G \mu(C_G(x))d\mu(x),$$

where $\mu(C_G(x)) = \int_G \chi_{C_2}(x,y)d\mu(y)$ and χ_{C_2} denotes the characteristic map of the set C_2.

Proof. Since $\mu(C_G(x)) = \int_G \chi_{C_2}(x,y)d\mu(y)$, we have by Fubini-Tonelli's Theorem:

$$\begin{aligned} cp(G) = (\mu \times \mu)(C_2) &= \int_{G\times G} \chi_{C_2} d(\mu \times \mu) \\ &= \int_G \int_G \chi_{C_2}(x,y)d\mu(x)d\mu(y) \\ &= \int_G \mu(C_G(x))d\mu(x). \end{aligned}$$ ♢

Lemma 2.2. *Let H be a subgroup of G of finite index. Then*

$$\mu(H) = [G : H]^{-1}.$$

Proof. Assume that $[G : H] = k$, where k is a positive integer. Then we have

$$1 = \mu(G) = \mu(\bigcup_{i=1}^{k} x_i H) = \sum_{i=1}^{k} \mu(x_i H) = k\mu(H). \qquad \diamondsuit$$

Let n be a positive integer. In the situation of the above lemma, we can easily see that if $[G : H] \geq n$, then $\mu(H) \leq 1/n$. At the same way if $[G : H] \leq n$, then $\mu(H) \geq 1/n$.

Lemma 2.3. *Let $G/Z(G)$ be a p-group of order p^r, where p is a prime and r is a positive integer. An element x belongs to $Z(G)$ if and only if $\mu(C_G(x)) > \frac{1}{p^{r-1}}$.*

Proof. It is clear that if $x \in Z(G)$ then $C_G(x) = G$ and therefore $\mu(C_G(x)) = 1 > \frac{1}{p^{r-1}}$. Conversely, assume that $\mu(C_G(x)) > \frac{1}{p^{r-1}}$ and $x \notin Z(G)$. Then, it is obvious that $[C_G(x) : Z(G)] \geq p$ and so we can see that

$$p^r = [G : Z(G)] = [G : C_G(x)][C_G(x) : Z(G)] \geq p[G : C_G(x)] .$$

Thus, $[G : C_G(x)] \leq p^{r-1}$ and it implies that $\mu(C_G(x)) \geq \frac{1}{p^{r-1}}$ by Lemma 2.2, which is a contradiction. Hence, $x \in Z(G)$ as required. $\diamondsuit$

Lemma 2.4. *Let $G/Z(G)$ be an elementary abelian p-group of rank 2, then*

$$cp_n(G) = \frac{p^2 + p - 1}{p^3},$$

for every prime p.

Proof. Assume that $G/Z(G)$ is an elementary abelian p-group of rank 2 . Then we may write G as the union of p^2 distinct cosets

$$G = Z(G) \cup x_1 Z(G) \cup x_2 Z(G) \cup \ldots \cup x_{p^2-1} Z(G)$$

and so $1 = \mu(G) = p^2 \mu(Z(G))$, since μ is a left Haar-measure.

If $a, b \in x_i Z(G)$, for $1 \leq i \leq p^2 - 1$, then $a = x_i z_1$ and $b = x_i z_2$ for some $z_1, z_2 \in Z(G)$ so that

$$ab = x_i z_1 x_i z_2 = x_i x_i z_1 z_2 = x_i x_i z_2 z_1 = x_i z_2 x_i z_1 = ba.$$

Thus, if $a \in x_iZ(G)$, then $C_G(a) = Z(G) \cup aZ(G) \cup a^2Z(G) \cup \ldots \cup a^{p-1}Z(G)$ and so

$$\begin{aligned}\mu(C_G(a)) &= \mu(Z(G)) + \mu(aZ(G)) + \mu(a^2Z(G)) + \ldots + \mu(a^{p-1}Z(G)) \\ &= p\mu(Z(G)) = p(\frac{1}{p^2}) = \frac{1}{p}.\end{aligned}$$

Thus, we have

$$\begin{aligned} cp(G) &= \int_G \mu(C_G(x))d\mu(x) \\ &= \int_{Z(G)} \mu(C_G(x))d\mu(x) + \sum_{i=1}^{p^2-1} \int_{x_iZ(G)} \mu(C_G(x))d\mu(x) \\ &= \mu(Z(G)) + \sum_{i=1}^{p^2-1} \tfrac{1}{p}\mu(x_iZ(G)) = (\tfrac{1}{p}(p^2-1)+1)\mu(Z(G)) \\ &= \tfrac{p^2+p-1}{p^3} \ . \end{aligned}$$ ♢

The following result has independent relevance, because it furnishes a bound for $cp_n(G)$.

Proposition 2.5. *If p is a prime and $G/Z(G)$ is a an elementary abelian p-group of rank 2, then*

$$cp_n(G) = \frac{p^n + p^{n-1} - 1}{p^{2n-1}}.$$

Proof. We may proceed by induction on n. By using Fubini-Tonelli theorem we can express $cp_n(G)$ as

$$\int_G \left[\int_{G^{n-1}} \chi_{C_{n-1}}(x_2, \ldots, x_n)\chi_{C_2}(x_1, x_2) \ldots \chi_{C_2}(x_1, x_n) d\mu^{n-1}(x_1, \ldots, x_n)\right] d\mu(x_1).$$

We shall integrate separately over $Z(G)$ and $G \setminus Z(G)$. For the integration over $Z(G)$ we can use the induction assumption, and get $\mu(Z(G))cp_{n-1}(G)$. The integration over $G \setminus Z(G)$ yeilds

$$\begin{aligned}&\int_{G-Z(G)} \left[\int_{C_G(x_1)^{n-1}} \chi_{C_{n-1}}(x_2, \ldots, x_n) d\mu^{n-1}(x_2, \ldots, x_n)\right] d\mu(x_1) \\ &= \mu(G - Z(G))\mu(C_G(x_1))^{n-1}\end{aligned}$$

Because x_1 commutes with all $x_2, \ldots, x_n$. Now, by summing both terms together we obtain

$$\frac{1}{p^2}\left(\frac{p^{n-1} + p^{n-2} - 1}{p^{2n-3}}\right) + (1 - \frac{1}{p^2})(\frac{1}{p})^{n-1} = \frac{p^n + p^{n-1} - 1}{p^{2n-1}}.$$ ♢

Lemma 2.6. *If p is a prime and $[G : Z(G)] = p^k$, then*

$$cp(G) \leq \frac{p^k + p - 1}{p^{k+1}},$$

for all integers $k \geq 2$.

Proof. Since $[G : Z(G)] = p^k$, so one can easily see that $\mu(Z(G)) = \frac{1}{p^k}$ and $\mu(C_G(a)) \leq \frac{1}{p}$, for all $a \in G \setminus Z(G)$ by Lemma 2.3. Now, by Lemma 2.1 we have

$$\begin{aligned} cp(G) &= \int_{Z(G)} \mu(C_G(x))d\mu(x) + \sum_{i=1}^{p^2-1} \int_{x_i Z(G)} \mu(C_G(x))d\mu(x) \\ &\leq \mu(Z(G)) + \sum_{i=1}^{p^k-1} \frac{1}{p}\mu(x_i Z(G)) \\ &= \frac{1}{p^k} + (p^k - 1)\frac{1}{p^{k+1}} = \frac{p^k + p - 1}{p^{k+1}}. \end{aligned}$$

◇

Proposition 2.7. *Let p be a prime and $k \geq 2$ be a positive integer. If the index $[G : Z(G)] = p^k$, then*

$$cp_n(G) \leq \frac{(p-1)\sum_{i=0}^{n-2} p^{i(k-1)} + p^{(n-1)k-n+2}}{p^{(n-1)k+1}}$$

for all integers $n \geq 2$. Furthermore, this bound is achieved if $G/Z(G)$ is an elementary abelian p-group of rank k and $[G : C_G(x)] = p$ for all $x \in G \setminus Z(G)$.

Proof. Suppose that $k \geq 2$ and we proceed by induction on n. If $n = 2$, then the proof is clear by Lemma 2.6. Now assume that the result holds for $n-1$, then by the hypothesis induction and the similar arguments as the proof of Proposition 2.5, we have

$$cp_{n-1}(G) = \mu(Z(G))cp_{n-1}(G) + (1 - \mu(Z(G))\mu(C_G(x_1))^{n-1}$$

$$\leq \frac{1}{p^k}\left(\frac{(p-1)\sum_{i=0}^{n-3} p^{i(k-1)} + p^{(n-2)k-n+3}}{p^{(n-2)k+1}}\right) + \frac{p^k - 1}{p^{n+k-1}}$$

$$= \frac{(p-1)\sum_{i=0}^{n-3} p^{i(k-1)} + p^{(n-2)k-n+3} + p^{(n-1)k-n+2} - p^{(n-2)k-n+2}}{p^{(n-1)k+1}}$$

$$= \frac{(p-1)\sum_{i=0}^{n-3} p^{i(k-1)} + p^{(n-2)k-n+2}(p-1) + p^{(n-1)k-n+2}}{p^{(n-1)k+1}}$$

$$= \frac{(p-1)\sum_{i=0}^{n-2} p^{i(k-1)} + p^{(n-1)k-n+2}}{p^{(n-1)k+1}}.$$

The second part of Proposition 2.7 comes from the fact that $\mu(C_G(a)) = \frac{1}{p}$, for all $a \in G \setminus Z(G)$. Hence we should have equality in all above relations and the proof is completed. ♢

3 Main Theorems

Proof of Theorem A. (i)⇒(ii). Assume that $G/Z(G)$ is an elementary abelian p-group of rank 2. Then Proposition 2.5 implies $cp_n(G) = \frac{p^n+p^{n-1}-1}{p^{2n-1}}$ and the statement follows.

(ii)⇒(i). Assume that $cp_n(G) = \dfrac{p^n + p^{n-1} - 1}{p^{2n-1}}$ and $G/Z(G)$ is not an elementary abelian p-group of rank 2. If $[G : Z(G)] \in \{1, p\}$, then $G/Z(G)$ is cyclic and so G is abelian which is a contradiction. Thus $[G : Z(G)] > p^2$ and therefore $\mu(Z(G)) < \frac{1}{p^2}$. Moreover, if $x \in G \setminus Z(G)$ then $\mu(C_G(x)) < \frac{1}{p}$ by Lemma 2.3. Thus

$$\begin{aligned} cp_n(G) &= \mu(Z(G))cp_{n-1}(G) + (1 - \mu(Z(G))[\mu(C_G(x_1))]^{n-1} \\ &< \mu(Z(G))\left(\frac{p^{n-1} + p^{n-2} - 1}{p^{2n-3}}\right) + (1 - \mu(Z(G)))(\frac{1}{p})^{n-1} \\ &< \frac{1}{p^2}\left(\frac{p^{n-1} + p^{n-2} - 1}{p^{2n-3}}\right) + (1 - \frac{1}{p^2})(\frac{1}{p})^{n-1} \\ &= \frac{p^n + p^{n-1} - 1}{p^{2n-1}} \end{aligned}$$

which is a contradiction and so (i) holds.

(i)⇒(iii). Assume that $G/Z(G)$ is an elementary abelian p-group of rank 2 and H is an extra-special p-group of order p^3. Thus $|H'| = |Z(H)| = |\Phi(H)| = p$ and this implies that $H/Z(H)$ is an elementary abelian p-group of rank 2 . Hence $G/Z(G)$ is isomorphic to $H/Z(H)$. Moreover, $|G'| = p$ by a famous Wiegold's bound (see [12, (3), vol.I, p.102]) and so G' is isomorphic to H'. Now, one can easily check that the diagram which appears in the definition of isoclinism between G and H is commutative. Hence G is isoclinic to H.

(iii)⇒(i). It is clear. ♢

Proof of Theorem B. (i)⇒(ii). If $G/Z(G)$ is an elementary abelian p-group of rank k, where $k = 2r$ and $r \geq 1$ is an integer, then by Proposition 2.5 and the fact that $\mu(Z(G)) = \frac{1}{p^k}$ and $\mu(C_G(x)) = \frac{1}{p}$, for all $x \in G \setminus Z(G)$ we have

$$\begin{aligned} cp_n(G) &= \mu(Z(G))cp_{n-1}(G) + (1 - \mu(Z(G))[\mu(C_G(x_1))]^{n-1} \\ &= \frac{1}{p^k}\left(\frac{(p-1)\sum_{i=0}^{n-3} p^{i(k-1)} + p^{(n-2)k-n+3}}{p^{(n-2)k+1}}\right) + \frac{p^k-1}{p^{n+k-1}} \\ &= \frac{(p-1)\sum_{i=0}^{n-2} p^{i(k-1)} + p^{(n-1)k-n+2}}{p^{(n-1)k+1}}. \end{aligned}$$

(ii)⇒(i). Since $[G : C_G(x)] = p$ for all $x \in G\backslash Z(G)$,

$$\mu(C_G(x)) = \frac{1}{p}.$$

Moreover, from the equalities

$$cp_n(G) = \mu(Z(G))cp_{n-1}(G) + (1 - \mu(Z(G)))[\mu(C(x_1))]^{n-1}$$

and

$$cp_n(G) = \frac{(p-1)\sum_{i=0}^{n-2} p^{i(k-1)} + p^{(n-1)k-n+2}}{p^{(n-1)k+1}}$$

we deduce that

$$\mu(Z(G)) = \frac{1}{p^k}.$$

Thus $[G : Z(G)] = p^k$. We should note that $G/Z(G)$ is an elementary abelian p-group, because for each noncentral element x of G, the subgroup $C_G(x)$ is normal of index p. So G modulo the intersection of these centralizers is elementary abelian.

(i)⇒(iii). Assume that $G/Z(G)$ is a p-elementary abelian group of rank $k = 2r$, where $r \geq 1$ is an integer and H is an extra-special p-group of order p^{k+1}. Thus $|H'| = |Z(H)| = |\Phi(H)| = p$ and this implies that $H/Z(H)$ is a p-elementary abelian of rank k. Hence $G/Z(G)$ is isomorphic to $H/Z(H)$. Now, we claim that $|G'| = p$.

For every $x \in G$ define the map

$$\varphi_x : t \in G \longmapsto \varphi_x(t) = [x, t] \in G'.$$

Since $G/Z(G)$ is abelian, G is nilpotent of class 2. Hence, we can easily see that φ_x is a homomorphism and $Ker\varphi_x = C_G(x)$. Moreover $G/Ker\varphi_x = G/C_G(x)$ is isomorphic to a subgroup I_x of G'. If $x \notin Z(G)$ then $|I_x| = p$ and so $p \leq |G'|$. If $|G'| > p$, then there exist elements $x, y \in G \backslash Z(G)$ such that $I_x \neq I_y$, $I_x = \langle a \rangle$ and $I_y = \langle b \rangle$. We may find the elements $u, v \in G$ such that $[x, u] = a$ and $[y, v] = b$. Thus we have $[x, v] \in I_x = \langle a \rangle$, $I_v = \langle b \rangle$ and so $[x, v] = 1$. Similarly, $[y, u] = 1$. Now it would imply that $[xy, u] = [x, u] = a$ and so $I_{xy} = \langle a \rangle$. Also, $[xy, v] = [y, v] = b$ and therefore $I_{xy} = \langle b \rangle$. This is a contradiction. Hence $|G'| = p$ and so G' is isomorphic to H'. Finally, by the same method as in the proof of Theorem A, we can show that the diagram which appears in the definition of isoclinism between G and H is commutative. Hence G and H are isoclinic.

(iii)⇒(i) It is clear. ♢

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