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AN IMPROVEMENT OF A BOUND OF GREEN

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A p–group G of order p^n (p prime, $n \geq 1$) satisfies a classic Green's bound $\log_p |M(G)| \leq \frac{1}{2}n(n-1)$ on the order of the Schur multiplier $M(G)$ of G. Ellis and Wiegold sharpened this restriction, proving that $\log_p |M(G)| \leq \frac{1}{2}(d-1)(n+m)$, where $|G'| = p^m$ ($m \geq 1$) and d is the minimal number of generators of G. The first author has recently shown that $\log_p |M(G)| \leq \frac{1}{2}(n+m-2)(n-m-1)+1$, improving not only Green's bound, but several other inequalities on $|M(G)|$ in literature. Our main results deal with estimations with respect to the bound of Ellis and Wiegold.

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1. Measuring the size of the Schur multiplier

All groups, which are considered in the present paper, are supposed to be finite. The literature of the last years on the groups of prime power order is becoming prominent in several areas, as described in [1] by Berkovich and Janko. They illustrate the importance of having a good classification for these groups, but, at the same time, different and complicated techniques are involved, once we want to proceed to a detailed study. Large part of the second volume of [1] is devoted to the case of 2–groups and this gives an idea of the difficulties which we may encounter.

A direction of research, widely investigated by Eick, Moravec and Leedham–Green in recent years, involves the notion of *coclass* and [3,4,5,9,10] describe those groups, for which restrictions of coclass are given. Instead in [2,6,7,12,13,14,15,18] it is privileged the notion of *Schur multiplier* and several analogies are possible among the techniques and the methods of all these papers.

Given a prime p, Schur [16] was probably the first who suggested to classify a p–group G, by looking at restrictions on its Schur multiplier $M(G)$ and a general bound, due to Jones [8, Theorem 3.1.4], has been generalized in [11], proving that

$$\log_p |M(G)| \leq \frac{1}{2}(n+m-2)(n-m-1)+1, \tag{1.1}$$

under the assumption that G has order p^n with derived subgroup G' of order p^m $(m, n \geq 1)$. For $m = 1$,

$$\log_p |M(G)| = \frac{1}{2}(n-1)(n-2)+1 \tag{1.2}$$

is achieved if and only if $G \simeq E \times Z$, where E is an extra–special p–group of order p^3 and exponent p and $Z = C_p \times \ldots \times C_p = C_p^{(n-3)}$ is an elementary abelian p–group of rank $n-3$. (1.1) improves a well–known result of Green (see [7, p.193, l.11] or [8, Corollary 3.1.5]), namely that

$$\log_p |M(G)| \leq \frac{1}{2}n(n-1). \tag{1.3}$$

There is a large literature, which is devoted to deduce structural information on G from the size of $|M(G)|$, and it is impossible to list here all the contributions. The reader may refer to [8] for further details. However we will compare (1.1) with [7, Theorem 2], which shows

$$\log_p |M(G)| \leq \frac{1}{2}(d-1)(n+m), \tag{1.4}$$

where $d = d(G)$ denotes the *minimal number of generators* of G. We recall that $G^{ab} = G/G'$ is the *abelianization* of G. Our first result is the following.

Theorem 1.1. *Let G be a p–group of $|G| = p^n$ and $|G'| = p^m$. Then*

$$\log_p |M(G)| \leq \begin{cases} \frac{1}{2}(d-1)(n+m), & if\ G^{ab} \simeq C_{p^{(n-m)/d}} \times \ldots \times C_{p^{(n-m)/d}}, \\ \frac{1}{2}(d-1)(n+m-1), & otherwise. \end{cases}$$

Theorem 1.1 gives not only a new proof of (1.4), but it also improves the bound when $G^{ab} \not\simeq C_{p^{(n-m)/d}} \times \ldots \times C_{p^{(n-m)/d}}$. The symbol $Frat(G)$ denotes the *Frattinian subgroup* of G, that is, the intersection of all maximal subgroups of G. For a p–group G, the condition $G' = Frat(G)$ is equivalent to require that G^{ab} is elementary abelian. Extra–special p–groups and several other classes of p–groups satisfy this condition. Our second main result is the following.

Theorem 1.2. *For a p–group G of $|G| = p^n$ and $|G'| = p^m$ the bound* (1.1) *is better than* (1.4) *for all $m \geq 1$, provided $G' = Frat(G)$.*

We will end the paper with some constructions related with [13,14,15]. Already Ellis and Wiegold [7] noted that there is an integer $t(G) \geq 0$, called *corank* of G, such that

$$\log_p |M(G)| = \frac{1}{2}n(n-1) - t(G) \tag{1.5}$$

and the idea of classifying p–groups by their corank is very intriguing.

For instance, if $t(G) = 0, 1$, then the structure of G is known by a result of Berkovich [1, Theorem 21.11, Vol. I]. For $t(G) = 2$, we find the description of Zhou [18]. The case $t(G) = 3$ was studied by Ellis in [6]. The case $t(G) = 4$ is in [2] and in general hypotheses in [15]. For $t(G) = 5$ we should look at [14]. In all these papers we have always a list of finitely many groups satysfying a restriction on the corank. Now the natural question is whether we could grow arbitrarily with $t(G)$. The answer, fortunately, is positive in the sense of [7, Corollary 4].

The origin of $t(G)$ is connected with (1.5) and Theorems 1.1 and 1.2 ensure that (1.1) is better than (1.4). This motivated us to propose a similar approach with (1.2) in [13], introducing an integer $s(G)$, which we call *generalized corank* of G, such that

$$\log_p |M(G)| = \frac{1}{2}(n-1)(n-2) + 1 - s(G). \tag{1.6}$$

For $s(G) = 0$, we mentioned that the structure of G is known. For $s(G) = 1, 2$ the first author gave contributions in [13]. For $s(G) \geq 3$, the problem is open. Then it is significant to ask whether an answer as in [7, Corollary 4] is possible in terms of $s(G)$ or not. We show that there exist families of p–groups of large values of generalized corank.

Theorem 1.3. *For each prime p, there exists a p–group G of exponent $2n$ such that $s(G) = 2(n-1)^2$, where $n \geq 1$. In particular, G has large generalized corank.*

2. Improvement of inequalities on the Schur multiplier

In the proof of the main theorems, we will use some notions of number theory and of low dimensional homology, which must be recalled from [1,8,11].

Lemma 2.1 (See [8], Theorem 2.2.10). *Given two p–groups G and H, the following condition is always satisfied*

$$M(G \times H) = M(G) \oplus M(H) \oplus (G^{ab} \otimes H^{ab}).$$

In particular,

$$\log_p |M(G \times H)| = \log_p |M(G)| + \log_p |M(H)| + \log_p |G^{ab} \otimes H^{ab}|.$$

The above splitting for the Schur multiplier of the direct product of p–groups is also known as Künneth Formula and is a crucial instrument in many proofs in literature. Another classical result is the following.

Lemma 2.2 (See [8], Theorem 2.5.5). *Let G be a p–group and N be a central subgroup of G. Then*

$$|M(G)|\ |G' \cap N| \leq |M(G/N)|\ |M(N)|\ |N \otimes G/NG'|.$$

Now we are going to adapt [8, Corollary 2.1.12] to one of the circumstances in Theorem 1.1.

Lemma 2.3. *If $G \simeq C_{p^{\alpha_1}} \times C_{p^{\alpha_2}} \times \ldots \times C_{p^{\alpha_d}}$ is an abelian p–group of $|G| = p^n$ such that $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_d$ are positive integers, then*

$$\log_p |M(G)| \leq \begin{cases} \frac{1}{2}n(d-1), & if\ \alpha_1 = \alpha_2 = \ldots = \alpha_d, \\ \\ \frac{1}{2}(n-1)(d-1), & otherwise. \end{cases}$$

Proof. Assume $\alpha_1 = \alpha_2 = \ldots = \alpha_d$. Then

$$G \simeq \underbrace{C_{p^{\alpha_1}} \times C_{p^{\alpha_2}} \times \ldots \times C_{p^{\alpha_d}}}_{d-\text{times}} = C^{(d)}_{p^{\frac{n}{d}}} \tag{2.1}$$

and by Lemma 2.1 we get

$$\log_p |M(G)| = \log_p |C_{p^{\frac{n}{d}}} \times C^{(2)}_{p^{\frac{n}{d}}} \times \ldots \times C^{(d-1)}_{p^{\frac{n}{d}}}| = \frac{n}{d} + 2\ \frac{n}{d} + \ldots + (d-1)\frac{n}{d} \tag{2.2}$$

$$= \frac{n}{d}\Big(1 + 2 + \ldots + (d-1)\Big) = \frac{n}{d}\Big(\frac{1}{2}(d-1)d\Big) = \frac{1}{2}n(d-1).$$

First we assume that $\alpha_1 = \alpha_2 = ... = \alpha_{d-1} > \alpha_d$. Hence for all $1 \leq i \leq d-1$ we should have $\alpha_i \geq \alpha_d + 1$ and so $n = \alpha_1 + \alpha_2 + \ldots + \alpha_{d-1} + \alpha_d \geq d\alpha_d + d - 1$, which implies $n - d + 1 \geq d\ \alpha_d$. From the last case and Lemma 2.1 we find

$$\log_p |M(G)| = \log_p |M(C_{p^{\alpha_1}} \times C_{p^{\alpha_2}} \times \ldots \times C_{p^{\alpha_{d-1}}}) \oplus M(C_{p^{\alpha_d}}) \oplus \tag{2.3}$$

$$(C_{p^{\alpha_2}} \times \ldots \times C_{p^{\alpha_{d-1}}}) \otimes C_{p^{\alpha_d}}|$$

and conclude that

$$\log_p |M(G)| = \frac{1}{2}(n-\alpha_d)(d-2) + (d-1)\alpha_d = \frac{1}{2}\Big(nd - 2n + d\alpha_d\Big). \tag{2.4}$$

Since $n - d + 1 \geq d\ \alpha_d$, we have

$$\log_p |M(G)| \leq \frac{1}{2}\Big(nd - 2n + d\ \frac{n-d+1}{d}\Big) = \frac{1}{2}(n-1)(d-1), \tag{2.5}$$

as required.

Now we may assume that there is an i such that $1 \leq i \leq d-2$ and $\alpha_i \neq \alpha_{i+1}$. We have $C_{p^{\alpha_1}} \times \ldots \times C_{p^{\alpha_{d-1}}}$ and, combining the induction hypothesis with Lemma 2.1, we find again (2.3),

$$\log_p |M(G)| = \log_p |M(C_{p^{\alpha_1}} \times C_{p^{\alpha_2}} \times \ldots \times C_{p^{\alpha_{d-1}}}) \oplus M(C_{p^{\alpha_d}}) \oplus \tag{2.6}$$

$$(C_{p^{\alpha_2}} \times \ldots \times C_{p^{\alpha_{d-1}}}) \otimes C_{p^{\alpha_d}}|.$$

from which we conclude in this case

$$\log_p |M(G)| = \frac{1}{2}(d-2)(n-\alpha_d-1) + (d-1)\alpha_d \tag{2.7}$$

$$= \frac{nd - d\alpha_d - d - 2n + 2\alpha_d + 2 + 2d\alpha_d - 2\alpha_d}{2} = \frac{1}{2}(nd + d\alpha_d - 2n - d + 2).$$

On the other hand, $\alpha_1 + \alpha_2 + \ldots + \alpha_d = n$ and $\alpha_1 \geq \alpha_{d-1}$ imply

$$\alpha_1 + \alpha_2 + \ldots + \alpha_d = n \geq (\alpha_d + 1) + \alpha_d + \ldots + \alpha_d = d\alpha_d + 1 \tag{2.8}$$

hence

$$\frac{n-1}{d} \geq \alpha_d. \tag{2.9}$$

Now (2.7) and (2.9) allow us to conclude that

$$\log_p |M(G)| = \frac{1}{2}\Big(nd + d\alpha_d - 2n - d + 2\Big) \leq \frac{1}{2}\Big(nd + d\frac{(n-1)}{d} - 2n - d + 2\Big) \tag{2.10}$$

$$= \frac{1}{2}(d-1)(n-1),$$

as claimed. □

The following observation shows one of the cases in which the bound of Lemma 2.3 is better than [7, Equation 2].

Remark 2.1. Assume that G is a p–group with minimal number of generators d and exponent e, satisfying $d \leq \lfloor \frac{n-1}{e-1} \rfloor - 1$. Then the bound of Lemma 2.3 is better than (1.4). For instance, this happens with $G = C_{p^2}^{(3)} \times C_p$, where (1.4) and [7, Equation 2] give the value 21/2 and 10, respectively, while the bound in Lemma 2.3 gives the value 9.

A partial version of Theorem 1.1, when the derived subgroup has prime order, is described by the next result.

Proposition 2.1. *If G is a p–group of $|G| = p^n$ with $|G'| = p$ and $G^{ab} \simeq C_{p^{\alpha_1}} \times \ldots \times C_{p^{\alpha_d}}$ for some positive integers $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_d$, then*

$$\log_p |M(G)| \leq \begin{cases} \frac{1}{2}(n+1)(d-1), \textit{if } \alpha_1 = \alpha_2 = \ldots = \alpha_d, \\ \\ \frac{1}{2}n(d-1), \qquad\qquad \textit{otherwise.} \end{cases}$$

Proof. Assume $\alpha_1 = \alpha_2 = \ldots = \alpha_d$. Since $|G'| = p$, $|M(G')| = 1$ and $|G' \otimes G^{ab}| = p^d$, Lemma 2.2 implies

$$|M(G)| \leq p^{-1} \; |M(G^{ab})| \; p^d, \tag{2.11}$$

but, applying Lemma 2.3 to G^{ab}, we get

$$\log_p |M(G^{ab})| \leq \frac{1}{2}(n-1)(d-1) \tag{2.12}$$

and so

$$\log_p |M(G)| \leq -1 + \frac{1}{2}(n-1)(d-1) + d = \frac{1}{2}(n+1)(d-1), \tag{2.13}$$

as claimed.

Now assume that there is an i such that $1 \leq i \leq d-2$ and $\alpha_i \neq \alpha_{i+1}$. Again we get (2.11), by Lemma 2.2. This time we apply Lemma 2.3 to (2.11) and get

$$\log_p |M(G^{ab})| \leq \frac{1}{2}(n-2)(d-1) \tag{2.14}$$

from which

$$\log_p |M(G)| \leq -1 + \frac{1}{2}(n-2)(d-1) + d = \frac{1}{2}n(d-1) \tag{2.15}$$

and the result follows. □

We have all the ingredients in order to prove Theorem 1.1.

Proof. [Proof of Theorem 1.1] We do induction on m. From Lemma 2.3, the result is true whenever G' is trivial. Assume G' is not trivial, $m \geq 1$ and T is a subgroup of $Z(G)$ of order p, contained in G'. Since $[G/T, G/T] = [G,G]T/T = G'T/T = G'/T$, we deduce $(G/T)^{ab} = (G/T)/(G'/T) \simeq G^{ab}$. This, jointly with $|M(T)| = 1$, allows us to apply Lemma 2.2, getting

$$|M(G)|\ |G' \cap T| \leq |M(T)|\ |M(G/T)|\ |T \otimes (G/T)^{ab}| = |M(G/T)|\ |T \otimes G^{ab}|. \tag{2.16}$$

Therefore

$$|M(G)| \leq |T \otimes G^{ab}||M(G/T)|p^{-1} = p^{d-1}\ |M(G/T)|. \tag{2.17}$$

The induction hypothesis implies

$$\log_p |M(G/T)| \leq \begin{cases} \frac{1}{2}(n-1+m-1)(d-1), & \text{if } G^{ab} \simeq C_{p^\alpha} \times \ldots \times C_{p^\alpha}, \\ \\ \frac{1}{2}(n-1+m-2)(d-1), & \text{otherwise.} \end{cases} \tag{2.18}$$

We may conclude in the first case that

$$\log_p |M(G)| \leq (d-1) + \frac{1}{2}(n-2+m)(d-1) = \frac{1}{2}(n+m)(d-1) \tag{2.19}$$

otherwise

$$\log_p |M(G)| \leq (d-1) + \frac{1}{2}(n-3+m)(d-1) = \frac{1}{2}(n+m-1)(d-1). \tag{2.20}$$

The result follows. □

The following consequence is a significant improvement of [7, Theorem 2].

Corollary 2.1. *Let G be a p–group of $|G| = p^n$ and $|G'| = p^m$ with $G^{ab} \not\simeq C_{p^\alpha} \times \ldots \times C_{p^\alpha}$ for some integer $\alpha \geq 1$. If $d = d(G)$ and $\delta = d(G/Z(G))$, then*

$$\log_p |M(G)| \leq \frac{1}{2}(d-1)(n-m-1) + (\delta-1)m - max\{0, \delta-2\}.$$

Proof. We follow [7, Proposition 1], defining $\overline{G} = G/Z(G)$, $\overline{x} = xZ(G) \in G/Z(G)$ and $[x, y]_\gamma = [x, y]\gamma_3(G) \in \gamma_2(G)/\gamma_3(G)$, where $\gamma_i(G)$ is the i–th term of the lower central series of G. The map

$$\psi : \overline{G}^{ab} \otimes \overline{G}^{ab} \otimes \overline{G}^{ab} \longrightarrow \frac{\gamma_2(G)}{\gamma_3(G)} \otimes \overline{G}^{ab} \tag{2.21}$$

$$\overline{x} \otimes \overline{y} \otimes \overline{z} \longmapsto \psi(\overline{x} \otimes \overline{y} \otimes \overline{z}) = ([x, y]_\gamma \otimes \overline{z}) + ([y, z]_\gamma \otimes \overline{x}) + ([z, x]_\gamma \otimes \overline{y})$$

turns out to be a homomorphism of groups (see [7, Proposition 1]) such that

$$|M(G)|\ |G'|\ |\mathrm{Im}\psi| \leq |M(G^{ab})|\ p^{\delta m}. \tag{2.22}$$

Since $|\mathrm{Im}\psi| \leq\ p^{max\{0,\delta-2\}}$, and $\log_p |M(G^{ab})| \leq \frac{1}{2}(d-1)(n-m-1)$ from Lemma 2.3, the result follows. □

A concrete example is listed below, where we evaluate (1.4) and (1.1). We also note that the following example illustrates that Corollary 2.1 is better than [7, Equation 2], whenever $d \leq \lfloor \frac{n-1}{e-1} \rfloor - 1$.

Example 2.1. By using GAP [17], the third group of order 16 in its library has $M(G) = C_2 \times C_2$, $G^{ab} = C_2 \times C_4$, $d(G) = 2$, $G' = C_2$ and $G/Z(G) = C_2 \times C_2$. Now the bound in Theorem 1.1 gives the value 2, while (1.4) gives the value 3.

We may prove Theorem 1.2.

Proof. [Proof of Theorem 1.2] It is straightforward to see that $-1 \geq -(n-m+1)$. On the other hand, $Frat(G)$ is characterized to be the set of all non–generators of G, then the set of generators of G and that of G/G' are the same and so $d = d(G) = d(G/Frat(G)) = d(G/G') = n - m$. This allows us to conclude that

$$\frac{1}{2}(n+m)(d-1) - 1 = \frac{1}{2}\Big((n+m)d - (n+m)\Big) - 1 \tag{2.23}$$

$$\geq \frac{1}{2}\Big((n+m)(n-m) - (n+m)\Big) - (n-m+1)$$

$$\geq \frac{1}{2}\Big((n+m)(n-m) - (n+m) - 2(n-m+1)\Big)$$

$$= \frac{1}{2}(n+m-2)(n-m-1).$$

Then

$$\frac{1}{2}(d-1)(n+m) \geq \frac{1}{2}(n+m-2)(n-m-1) + 1 \geq \log_p |M(G)|, \tag{2.24}$$

as claimed. □

3. On the size of the generalized corank

The importance of (1.1) is emphasized by Theorems 1.1 and 1.2. This justifies the interest for classifications of p–groups by their generalized corank. The next lemma shows a rule, similar to that in Lemma 2.1, when we want to compute the generalized corank of the direct product of two groups.

Lemma 3.1. *A p–group G of order p^n and a p–group H of order p^m satisfy the condition*

$$s(G \times H) = s(G) + s(H) + (nm - (2 + \log_p |G^{ab} \otimes H^{ab}|)).$$

Proof. Since $|G \times H| = |G| \cdot |H| = p^{n+m}$, (1.6) becomes

$$\log_p |M(G \times H)| = \frac{1}{2}(n + m - 1)(n + m - 2) + 1 - s(G \times H) \tag{3.1}$$

$$= \frac{1}{2}((n^2 - 3n + 2) + (m^2 - 3m + 2) + 2nm) - s(G \times H)$$

$$= \frac{1}{2}(n - 1)(n - 2) + \frac{1}{2}(m - 1)(m - 2) + nm - s(G \times H).$$

Separately,

$$\log_p |M(G)| + \log_p |M(H)| + \log_p |G^{ab} \otimes H^{ab}| \tag{3.2}$$

$$= \frac{1}{2}(n - 1)(n - 2) + 1 - s(G) + \frac{1}{2}(m - 1)(m - 2) + 1 - s(H) + \log_p |G^{ab} \otimes H^{ab}|$$

Now Lemma 2.1 implies the equality

$$\frac{1}{2}(n - 1)(n - 2) + \frac{1}{2}(m - 1)(m - 2) + nm - s(G \times H) \tag{3.3}$$

$$= \frac{1}{2}(n - 1)(n - 2) + 1 - s(G) + \frac{1}{2}(m - 1)(m - 2) + 1 - s(H) + \log_p |G^{ab} \otimes H^{ab}|$$

which becomes

$$s(G) + s(H) + nm = s(G \times H) + 2 + \log_p |G^{ab} \otimes H^{ab}| \tag{3.4}$$

that is

$$s(G \times H) = s(G) + s(H) + (nm - (2 + \log_p |G^{ab} \otimes H^{ab}|)). \tag{3.5}$$

The result follows. □

We have the following consequence for cyclic p–groups.

Corollary 3.1. *If $\gcd(p^m, p^n) = p^d$ with $d \geq 1$, then*

$$s(C_{p^m} \times C_{p^n}) = \frac{1}{2}\Big((n + m)^2 - (3(n + m) + 2d) + 4\Big).$$

In particular, if $m = n$, then $s(C_{p^n} \times C_{p^n}) = 2(n-1)^2$.

Proof. Denoting $H = C_{p^n}$ and $K = C_{p^m}$, we have $s(H) = \frac{1}{2}(n-1)(n-2)+1$, $s(K) = \frac{1}{2}(m-1)(m-2)+1$ and $|H^{ab} \otimes K^{ab}| = |H \otimes K| = |C_{p^n} \otimes C_{p^m}| = |C_{p^d}| = p^d$. Lemma 2.1 implies

$$s(H \times K) = \frac{1}{2}(n-1)(n-2)+1+\frac{1}{2}(m-1)(m-2)+1+nm-2-d \tag{3.6}$$

$$= \frac{1}{2}\Big((n-1)(n-2)+(m-1)(m-2)+2nm-2d\Big)$$

$$= \frac{1}{2}(n^2-3n+2+m^2-3m+2+2nm-2d) = \frac{1}{2}\Big((n+m)^2-(3(n+m)+2d)+4\Big).$$

In particular, if $m = n$, then $d = n$ and the value of $s(C_{p^n} \times C_{p^n})$ follows. $\square$

Now a large part of Theorem 1.3 can be proved.

Proof. [Proof of Theorem 1.3] From Corollary 3.1 we consider $G = H \times H$, for a cyclic p–group $H = C_{p^n}$, and have $s(G) = 2(n-1)^2$, which is a strictly increasing function for all $n \geq 1$. Note that G has exponent $2n$. The result follows. $\square$

Theorem 1.3 does not show that there exists a p–group with arbitrary prescribed value of generalized corank for each positive integer. This can be deduced only for values of the generalized corank of the form $2(n-1)^2$ for $n \geq 1$. What we can deduce from Theorem 1.3 is that the value of $s(G)$ may be large.

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