SOME OPEN QUESTIONS ON A RESULT OF B.H. NEUMANN

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Abstract. A subgroup K of a group G is called almost normal in G if it has finitely many conjugates in G. The influence of these subgroups is strong on the group structure. Indeed, B.H. Neumann proves in the 1955 that $|G : Z(G)|$ is finite if and only if each K is almost normal in G. Many authors have successively generalized this result and the present survey makes the point of the situation, illustrating a new perspective for wider generalizations.

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1. A brief overview

In [1] R. Baer describes the structure of the groups with finite conjugacy classes, or FC-*groups*. Some years later B.H. Neumann writes the two papers [34] and [35] which will be classic works in the theory of FC-groups. See also [41, Vol.I, §4.3]. It is a common opinion that [1, 34, 35] introduce a new approach of study of the infinite groups. Some results, which originated from [1, 34, 35], are in [2, 7, 8, 9, 10, 14, 17, 18, 19, 21, 22, 23, 24, 25, 26, 28, 29, 30, 31, 32, 36, 38, 39, 40, 42, 43, 44, 45, 46, 47, 48, 49]. The list is very partial and reflects only some topics which we will illustrate successively.

The investigations in [34] and [35] differ from those in [1] for two main motivations. The first deals with the bounds of the finite conjugacy classes (see [41, Theorem 4.35] or [34, Theorem 3.1]). The second deals with the covering properties of a group by means of suitable subgroups (see [41, Theorem 4.16, Lemma 4.17]). More precisely, [41, Theorem 4.35] states that a group G has finite index $|G : Z(G)|$ if and only if the conjugacy classes of G are finite and bounded, or equivalently, if and only if G' is finite. [41, Theorem 4.16] states that $|G : Z(G)|$ is finite if and only if G has a finite covering consisting of abelian subgroups. It is clear the connection among the theory of the coverings of subgroups and that of

FC-groups. Variations on these themes have interested many authors in different contexts in the last years. See for instance [5, 6, 7, 8, 9, 10, 17, 18, 19].

2. Anti-XC-groups and Neumann's Theorem

Assume from now that $\mathfrak{X}$ denotes an arbitrary class of groups which is closed with respect to forming subgroups and quotients, $\mathfrak{F}$ is the class of all finite groups, $\mathfrak{F}_\pi$ is the class of all finite π-groups (π set of primes), $\check{\mathfrak{C}}$ is the class of all Chernikov groups, $\mathfrak{PF}$ is the class of all polycyclic-by-finite groups, $\mathfrak{S}_2\mathfrak{F}$ is the class of all (soluble minimax)-by-finite groups. It is easy to check that

$$\mathfrak{F} \subseteq \check{\mathfrak{C}} \subseteq \mathfrak{S}_2\mathfrak{F}, \quad \mathfrak{F} \subseteq \mathfrak{PF} \subseteq \mathfrak{S}_2\mathfrak{F}, \quad \check{\mathfrak{C}} \cap \mathfrak{PF} = \mathfrak{F}.$$

See [30, 31, 32, 41, 42] for details.

Given a positive integer r and a group G, we recall that the operator $\mathbf{L}$, defined by

$$\mathbf{L}\mathfrak{X} = \{G \mid \langle g_1, g_2, \ldots, g_r\rangle \in \mathfrak{X}, \forall g_1, g_2, \ldots g_r \in G\},$$

from $\mathfrak{X}$ to $\mathfrak{X}$ is called *local operator* for $\mathfrak{X}$. See [31, §C, p.54]. We recall that the operator $\mathbf{H}$, which associates to $\mathfrak{X}$ the class of *hyper-$\mathfrak{X}$-groups* is called *extension operator*. See [31, §E, p.60]. Notations and terminology follow [30, 31, 32, 41, 42].

As already recalled in the abstract, a subgroup K of a group G is called *almost normal* in G if K has finitely many conjugates in G, that is, if $|G : N_G(K)|$ is finite. Neumann's Theorem [41, Chapter 4, Vol.I, p.127] shows that G has each K which is almost normal in G if and only if $G/Z(G) \in \mathfrak{F}$. We have

$$N_G(Cl_G(K)) = core_G(N_G(K)) = \bigcap_{x\in G} N_G(K)^x = \bigcap_{x\in G} N_G(K^x),$$

where $Cl_G(K)$ is the set of conjugates of K in G. $|G : N_G(K)| = |Cl_G(K)|$ is finite if and only if $G/core_G(N_G(K)) \in \mathfrak{F}$. In [25, 26] G has *$\mathfrak{F}$-classes of conjugate subgroups*, if $G/core_G(N_G(K)) \in \mathfrak{F}$ for each K in G.

Thus Neumann's Theorem can be reformulated as follows.

Theorem 2.1 (Neumann's Theorem). *A group G has $\mathfrak{F}$-classes of conjugate subgroups if and only if $G/Z(G) \in \mathfrak{F}$.*

See [26, Introduction]. More generally, G has $\mathfrak{X}$-*classes of conjugate subgroups*, if $G/core_G(N_G(K)) \in \mathfrak{X}$ for each K in G. In this context there are two questions of great interest.

Open Question 2.2. For which choice of $\mathfrak{X}$, in a group G the condition to have $\mathfrak{X}$-classes of conjugate subgroups is equivalent to $G/Z(G) \in \mathfrak{X}$?

Theorem 2.1 answers positively Question 2.2 for $\mathfrak{X} = \mathfrak{F}$. We know a positive answer of Question 2.2 also for $\mathfrak{X} = \mathfrak{PF}$ from [25, Main Theorem]. Indeed, this result states that a group G has $\mathfrak{PF}$-classes of conjugate subgroups if and only if $G/Z(G) \in \mathfrak{PF}$. Unfortunately, Question 2.2 has a negative answer for $\mathfrak{X} = \check{\mathfrak{C}}$. [26, Main Theorem] describes groups having $\check{\mathfrak{C}}$-classes of conjugate subgroups and [26, Section 4] shows an example of a group having $\check{\mathfrak{C}}$-classes of conjugate subgroups with $G/Z(G) \notin \check{\mathfrak{C}}$. Therefore Question 2.2 should be strengthened as follows.

Open Question 2.3. What it the structure of a group having $\mathfrak{X}$-classes of conjugate subgroups?

Question 2.3 is partially answered in [44, Main Theorem], where there is a description of the groups having $\mathfrak{S}_2\mathfrak{F}$-classes of conjugate subgroups. Here some restrictions are done and so Question 2.2 could be answered positively. This is an open problem.

Recall that

$$Z_{\mathfrak{X}}(G) = \{x \in G \mid G/C_G(\langle x \rangle^G) \in \mathfrak{X}\}$$

is a characteristic subgroup of G, called *XC-center* of G. See [31, Definition B.1, Proposition B.2]. G is called *XC-group* if it coincides with its *XC-center*. *FC-groups*, *CC-groups*, *PC-groups* and *MC-groups* are obtained when we consider respectively $\mathfrak{F}$, $\check{\mathfrak{C}}$, $\mathfrak{P}\mathfrak{F}$, $\mathfrak{S}_2\mathfrak{F}$. These are studied in [2, 14, 28, 30, 31, 32, 36, 38, 39, 40, 42, 43, 44, 45, 46, 47]. Finally, G is called *HXC-group* if $G = Z_{\mathbf{H}\mathfrak{X}}(G)$.

If G has $\mathfrak{F}$-classes of conjugate subgroups, then it is an FC-group. From [26, Lemma 2.3], if G has $\check{\mathfrak{C}}$-classes of conjugate subgroups, then it is a CC-group. From [25, Corollary 2.7], if G has $\mathfrak{P}\mathfrak{F}$-classes of conjugate subgroups, then it is a PC-group. From [44, Lemma 2.4], if G has $\mathfrak{S}_2\mathfrak{F}$-classes of conjugate subgroups, then it is an MC-group. These facts can be generalized in the next form.

Lemma 2.4. *(*[46, Lemma 2.1]*) Assume that $\mathfrak{F}\mathfrak{X} = \mathfrak{X}$. If G has $\mathfrak{X}$-classes of conjugate subgroups, then $Z_{\mathfrak{X}}(G) = G$.*

We recall that $\mathfrak{X}$ is called *Dietzmann class*, if for every group G and $x \in G$, the following implication is true:

$$(*) \quad \text{if } x \in Z_{\mathfrak{X}}(G) \text{ and } \langle x \rangle \in \mathfrak{X}, \text{ then } \langle x \rangle^G \in \mathfrak{X}.$$

See [31, Definitions B.1 and B.6] or [11]. Dietzmann classes are studied in [30, 31, 32, 42]. FC-groups form a Dietzmann class as we note in [31, Proposition D.3, b)]. In particular, this is true for periodic PC-groups, which are obviously FC-groups. Note that $\mathfrak{F}$ is a Dietzmann class (se [31, Proposition B.7, b)]) but $\mathfrak{P}\mathfrak{F}$ is not a Dietzmann class (see [31, Example B.8, c)]). Unfortunately, it is not known whether PC-groups, CC-groups or MC-groups form a Dietzmann class. See always [30, 31, 32, 42]. But it is easy to check that PC-groups, CC-groups or MC-groups extend locally the class of FC-groups. Therefore, the next result is significant.

Theorem 2.5. *(*[31, Theorem E.3]*) If $\mathfrak{F}_\pi \subseteq \mathfrak{X} \subseteq \mathbf{L}\mathfrak{F}_\pi$, then the HXC-groups form a Dietzmann class.*

From Lemma 2.4 and Theorem 2.5, it is meaningful to ask whether we may strengthen Neumann's Theorem, considering the following property:

$(**)$ if K is a non-finitely generated subgroup of a group G,
then $G/core_G(N_G(K)) \in \mathfrak{X}$, where $\mathfrak{F}_\pi \subseteq \mathfrak{X} \subseteq \mathbf{L}\mathfrak{F}_\pi$.

G is called *anti-XC-group* if it satisfies $(**)$. They are studied in [46]. *Anti-FC-groups* are described in [12]. *Anti-CC-groups* and *anti-PC-groups* are described in [45]. We cannot forget in this line of research [20], whose methods are used both in [12] and [20]. On another hand, the ideas and the methods go back to [33] and deal with the structure of groups with given properties of a system of subgroups. Among the impressive literature in this topic, we mention [3, 4, 13, 15, 16, 21, 22, 23, 24, 27, 29, 37, 51].

3. Locally finite case

Following [12, 20, 45], in this Section we will give a brief description of the locally finite groups satisfying $(**)$. They are discussed in [46]. The considerations in Section 2 allow us to prove easily the next two results.

Lemma 3.1. *Subgroups and quotients of anti-XC-groups are anti-XC-groups.*

Lemma 3.2. *If G is an anti-XC-group and $Z_{\mathfrak{X}}(G) = G$, then G has $\mathfrak{X}$-classes of conjugate subgroups.*

Overlapping [45, Lemma 3.3] and from Lemmas 3.1 and 3.2, we have as follows.

Lemma 3.3. *Assume that x is an element of the anti-XC-group G. If $A = Dr_{i\in I}A_i$ is a subgroup of G consisting of $\langle x\rangle$-invariant nontrivial direct factors A_i, $i \in I$, with infinite index set I, then x belongs to $Z_{\mathfrak{X}}(G)$.*

Lemma 3.3 has the next consequence, which is straightforward.

Corollary 3.4. *Assume that G is an anti-XC-group and $A = Dr_{i\in I}A_i$ is a subgroup of G consisting of infinitely many nontrivial direct factors. Then A is contained in $Z_{\mathfrak{X}}(G)$.*

The next lemma overlaps [45, Lemma 3.7].

Lemma 3.5. *Assume that g is an element of the anti-XC-group G and $A = Dr_{i\in I}A_i$ is a subgroup of G, with I as in Lemma 2.3. If $g \in N_G(A)$ and $g^n \in C_G(A)$ for some positive integer n, then g belongs to $Z_{\mathfrak{X}}(G)$.*

Combining the above Lemmas 3.1, 3.2, 3.3, 3.5 and Corollary 3.4 we get the next corollary, whose proof overlaps [45, Corollary 3.9].

Corollary 3.6. *If the anti-XC-group G has an abelian torsion subgroup that does not satisfy the minimal condition on its subgroups, then all elements of finite order belong to $Z_{\mathfrak{X}}(G)$.*

All the above considerations allow us to describe the locally finite case.

Theorem 3.7. *If G is a locally finite anti-XC-group, then either G has $\mathfrak{X}$-classes of conjugate subgroups or G is a Chernikov group.*

Proof. From Lemmas 3.1, 3.2, 3.3, 3.5 and Corollaries 3.4, 3.6, we may argue as in [45, Theorem 3.12, Proof], considering $\mathfrak{X}$, $Z_{\mathfrak{X}}(G)$ and the result follows. □

Theorem 3.7 extends [45, Theorems 3.11 and 3.12] and similar situations in [12, 20]. The locally nilpotent case can be treated in an analogous way, invoking some results in [50].

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