THE PACKAGE OSB 1.0 FOR DECOMPOSING B^1-GROUPS

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Abstract. The notion of tent allows us to have an efficacious representation of Butler groups and a detailed description of decompositions of B^1-groups. The present paper deals with the package OSB 1.0, which allows us to optimize some algorithmic criteria, which are involved in the study of the tents and, more generally, in that of the B^1-groups. Some significant examples are shown.

Keywords: Decompositions of B^1-groups; tent; algorithms for Butler groups; OSB 1.0.

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1. Feedback and Motivations

The literature on Butler groups is relatively recent, since Butler groups were introduced in [1]. Nowadays there are mainly two approaches in studying the topic: the first has been introduced by D. M. Arnold and C. Vinsonhaler (see [2]) and the second has been introduced by C. De Vivo and C. Metelli (see [3, 4, 5, 6]). We refer to the second approach in the present paper, assuming that the reader has familiarity with notations and terminology of [3, 4, 5, 6].

A first motivation in writing the present paper is that recent results in [7, 8, 9] are not available in English. Most has been communicated here for a wider audience. A second motivation is the absence of packages and literature in algorithmic problems for Butler groups. We noted this fact, after using standard packages as GAP.

The results of Section 2 allow us to describe OSB 1.0, finding new examples for open questions in [2, 3, 4, 5, 6, 8, 9]. Unfortunately, we were not able to solve such questions, which remain still open to the best of our knowledge.

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2. Admissibility and Indecomposability

The present Section recalls some conditions of admissibility and indecomposability for B^1-groups. An m-tuple $\mathcal{E} = (b_{E_1}, \ldots, b_{E_m})$ of 2-pointed partitions of $I = \{1, \ldots, m\}$ is *admissible* if $\mathcal{E}$ is a redundant base of the $\mathbb{Z}_2$-vector space $\mathbb{B}(m)$ of 2-partitions of I, that is, $\mathcal{E}$ is an automorphism of $\mathbb{B}(m)$, written with respect to the standard admissible m-tuple $(p_1, \ldots, p_m)$ of pointed 2-partitions of I. Most of the proofs can be found in [8, Section 2], [9, Sections 2, 3], and [3, 4, 5] so they have been omitted. These results are fundamental for describing the package OSB 1.0. The following result recalls [9, Proposizione 2.1.6] and is in [5].

Proposition 2.1 (see [5]). *There exists a base change $\mathcal{E}$ between two tents $\mathbf{T}$ and $\mathbf{T_1}$ if and only if*

$$\mathbf{T_1} = \mathcal{S}_1(\mathcal{S}_2(\ldots(\mathcal{S}_{k-1}(\mathcal{S}_k(\mathbf{T}))), \tag{2.1}$$

where $\mathcal{S}_1, \ldots, \mathcal{S}_k$ are exchanges such that $\mathcal{E} = \mathcal{S}_1 \ldots \mathcal{S}_k$ and

$$\mathbf{T} \subseteq D(\mathcal{S}_k), \mathcal{S}_k(\mathbf{T}) \subseteq D(\mathcal{S}_{k-1}), \ldots, \mathcal{S}_2(\ldots(\mathcal{S}_k(\mathbf{T})) \subseteq D(\mathcal{S}_1). \tag{2.2}$$

Note that the notion of *tent* $\mathbf{T}$, that of *base change* $\mathcal{E}$ between two tents $\mathbf{T}$ and $\mathbf{T_1}$, in particular that of *exchange* $\mathcal{S}$ and that of *domain* $D(\mathcal{E})$, which are used in Proposition 2.1, follow [3, 4, 5]. The next result is in [9, Proposizione 3.1.5].

Proposition 2.2 (see [9]). *There exist two m-tuples $\mathcal{E}$ and $\mathcal{F}$ of 2-partitions of $I = \{1, \ldots, m\}$ such that*

(1) *$\mathcal{E}$ is not admissible,*
(2) *$\mathcal{E} * \mathcal{F} = (p_1, \ldots, p_m)$;*

if and only if there exists a k-tuple $\mathcal{E}'$ of 2-partitions of $\{1, \ldots, n\}$, where $k < n \leq m$, and an n-tuple $\mathcal{F}'$ of 2-partitions of $\{1, \ldots, k\}$ such that

$(1)'$ *$\mathcal{E}' * \mathcal{F}' = (p_1, \ldots, p_n)$;*
$(2)'$ *$\mathcal{E}'$ is linearly independent;*
$(3)'$ *there are no components of $\mathcal{E}'$ which are pointed 2-partitions.*

Note that we may find in [3, 4, 5] the notion of *admissibility* and the notion of product $\mathcal{E} * \mathcal{F}$ between two m-tuples $\mathcal{E}$ and $\mathcal{F}$, which have been used in Proposition 2.2. The following result is in [3].

Proposition 2.3 (see [3]). *If $D(\mathcal{E})$ is indecomposable, then*

$$\mathcal{E} * \mathcal{E}^{-1} = (p_1, \ldots, p_m). \tag{2.3}$$

The notion of *indecomposability* of $D(\mathcal{E})$ can be found in [3, 4, 5]. Note that [9, Esempio 3.2.2] shows that the converse of Proposition 2.3 cannot hold. The following result is in [9, Proposizione 3.2.5].

Proposition 2.4 (see [9]). *Let $\mathcal{E}$ be an automorphism of $\mathbb{B}(m)$ such that*

- $\mathcal{E} * \mathcal{E}^{-1} = (p_1, \dots, p_m)$,
- $\mathcal{E}^{-1} * \mathcal{E} = (p_1, \dots, p_m)$.

Assume that $A \in D(\mathcal{E})$ and $A^{-1} = \{i_1, \dots, i_s\}$. Deleting A, we get an admissible s-tuple $\mathcal{E}'$ of 2-partitions of A^{-1} such that:

- $\mathcal{E}' * (\mathcal{E}')^{-1} = (p_{i_1}, \dots, p_{i_s})$,
- $(\mathcal{E}')^{-1} * \mathcal{E}' = (p_{i_1}, \dots, p_{i_s})$,
- $D(\mathcal{E}') = \{B \cap \{i_1, \dots, i_s\} \,|\, B \in D(\mathcal{E}) \text{ and } A \subseteq B\}$.

The following formulation of Proposition 2.4 is more adapt for algorithmic treatments. This is unpublished and we refer to [9, 3, 4, 5] for the notion of *closed-chain of n-level* and for the notion

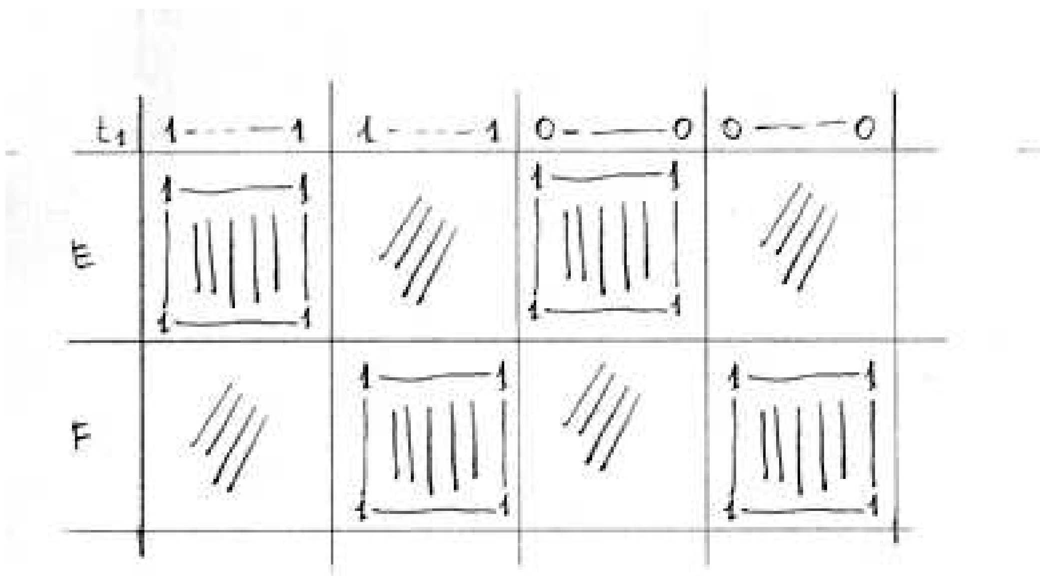
$$(part_t(t_1), \dots, part_t(t_m))$$

of *minimal partitions* of a tent $\mathbf{T}$ with base types $(t_1, \dots, t_m)$.

Proposition 2.5. *Let $\mathcal{E}$ be an automorphism of $\mathbb{B}(m)$ minimal with respect to the following conditions:*

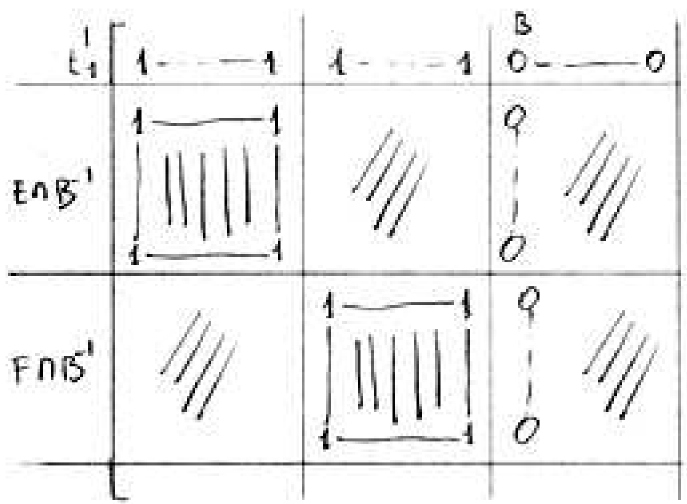
(i) $\mathcal{E} * \mathcal{E}^{-1} = \mathcal{E}^{-1} * \mathcal{E} = (p_1, \dots, p_m)$,
(ii) *$D(\mathcal{E})$ is decomposable.*

Then $D(\mathcal{E}) = \mathbf{T}$ is described by the following diagram up to a permutation of I,



where $\{\{1\}, E, F\}$ is a 3-partition of I.

Proof. Since $D(\mathcal{E}) = \mathbf{T}$ is decomposable, we may assume, up to a permutation of I, that $part_t(t_1) \leq \{\{1\},E,F\}$, where $\{\{1\}, E, F\}$ is a 3-partition of I. Then each $A \in D(\mathcal{E})$ such that $1 \in A$, contains either E or F. Assume that there is $B \in D(\mathcal{E})$ such that $1 \notin B$ and $B^{-1} \cap E \neq \emptyset$ and $B^{-1} \cap F \neq \emptyset$. Proposition 2.4 implies that if we cancel B and those 2-partitions b_{E_j} greater or equal than p_B, we get an s-tuple $\mathcal{E}' = (b_{E'_j} | j \in B^{-1})$ which is admissible for B^{-1}. Note that $s = | B^{-1} | < m$ and the domain of $\mathcal{E}'$ is $\mathbf{T}' = D(\mathcal{E}')$. This can be obtained by $\mathbf{T} = D(\mathcal{E})$ canceling B and those sets $C \in D(\mathcal{E})$ such that $B \nsubseteq C$ and $\mathcal{E}' * (\mathcal{E}')^{-1} = (\mathcal{E}')^{-1} * \mathcal{E}' = (p_j | j \in B^{-1})$. Since $\mathcal{E}$ is minimal, $D(\mathcal{E}')$ is indecomposable. The way in which we constructed $D(\mathcal{E}')$ from $D(\mathcal{E})$ implies that $D(\mathcal{E}')$ is described by the following diagram:



Therefore, $part_{t'}(t'_i) \leq \{(1)(B^{-1} \cap E)(B^{-1} \cap F)\}$, and so t'_1 is not uniquely given by $D(\mathcal{E}')$. Then $D(\mathcal{E}')$ is decomposable. This is a contradiction and so the result follows. □

An easy consequence of Proposition 2.5 is that $D(\mathcal{E}^{-1})$ has the same diagram of $D(\mathcal{E})$, up to a permutation of I.

3. Description of OSB 1.0

The algorithmic criteria in previous Section 2 can be translated in terms of a package for computer: this is the idea behind OSB 1.0. Its description is the main goal of the present Section. OSB 1.0 can execute 4 processes as follows.

Process 1. **Admissibility**. This is related to Proposition 2.2. OSB 1.0 checks if an m-tuple $\mathcal{E}$ of $\mathbb{B}(m)$ is admissible. If $\mathcal{E}$ is admissible, we may get $\mathcal{E}^{-1}$.

Step 1. Insert $\mathcal{E}$.
Step 2. Transform $\mathcal{E}$ in an $m \times m$ matrix over $\mathbb{Z}_2$.
Step 3. Calculate the combinations without repetitions of m elements on n places with n running in I.
Step 4. From Step 3, sum the columns in the place of the combination.
Step 5. There are all the sums between columns.
Step 6. Consider the sums which are related to pointed partitions: $\mathcal{E}^{-1}$ follows easily.

Further details on the architecture of Process 1 can be found in [7]. We use in the next process the notion of redundant bases $\mathcal{E} = (b_{E_1}, \ldots, b_{E_m})$ and $\mathcal{E}' = (b_{F_1}, \ldots, b_{F_m})$, the notion of *inf* and *sup* of a set of partitions of I and the notion of partitions

$$\mathcal{A}_i = inf\{b_{E_j} | j \in F_i\}, \mathcal{C}_i = inf\{b_{E_j} | j \in F_i^{-1}\}, \tag{3.1}$$

that work out

$$\mathcal{E} * \mathcal{E}' = (\mathcal{A}_1 \ sup \ \mathcal{C}_1, \ldots, \mathcal{A}_m \ sup \ \mathcal{C}_m). \tag{3.2}$$

These notions can be found in [3, 4, 5] and in [9, §1.0].

Process 2. **Product**. This is related to Propositions 2.2, 2.3, 2.4, 2.5. Here OSB 1.0 calculates the product $\mathcal{E} * \mathcal{F}$ of two m-tuples of $\mathbb{B}(m)$.

Step 1. Insert two bases $\mathcal{E}$ and $\mathcal{E}'$.
Step 2. Transform $\mathcal{A}_i$ and $\mathcal{C}_i$ in $m \times m$ matrices over $\mathbb{Z}_2$.
Step 3. Fix the i-th column of $\mathcal{E}'$ and do inf of the columns the matrix of $\mathcal{E}$ looking at the 1s on the fixed column.
Step 4. $\mathcal{A}_i$ is given.
Step 5. Do Step 3 for 0s instead of 1s.
Step 6. $\mathcal{C}_i$ is given.

Further details on the architecture of Process 2 can be found in [7]. We use in the next process the notion of *block*, *of length* of a block and that of *sub-block* $\mathcal{U}$ of a partition of I.

Process 3. **Domain**. This is related to Propositions 2.1, 2.4, 2.5.

Step 1. Apply Process 2 as above and get $\mathcal{A}_i$, $\mathcal{C}_i$.

Step 2. If $\mathcal{A}_i$ or $\mathcal{C}_i$ has a block of length ≤ 2, then take the column.
Step 3. If $\mathcal{A}_i$ or $\mathcal{C}_i$ has a block of length > 2, then calculate all the combinations of the elements in the column.
Step 4. For each sub-block $\mathcal{U}$ do a columns of 0s of length m and control if such a sub-block lies in some $\mathcal{A}_i$ or $\mathcal{C}_i$.
Step 5. If $\mathcal{U}$ lies in in some $\mathcal{A}_i$ or $\mathcal{C}_i$, then put 1s in the i-th place of the column as in Step 4.
Step 6. After checking $\mathcal{A}_i$ or $\mathcal{C}_i$ as in Steps 2-5, delete, if there are 0s, if the column already exists, if the column is different from the column which has 0s on the elements of $\mathcal{U}$ and 1s elsewhere.
Step 7. After checking $\mathcal{A}_i$ or $\mathcal{C}_i$ as in Steps 2-5, add the column to the domain, if Step 6 is false.

Further details on the architecture of Process 3 can be found in [7].

Process 4. **Decomposition in exchanges.** This process is related to Proposition 2.1 and its description is very similar to those of the previous processes. Details can be found in [7].

Remark 3.1. *Combining Processes 1, 2, 3, 4, OSB 1.0 gives detailed information for an arbitrary decomposition of a* $\mathbb{B}(m)$ *in the sense of Propositions 2.1, 2.2, 2.3, 2.4, 2.5.*

Remark 3.2. *Combining Processes 1, 2, 3, 4, and Proposition 2.1 we may get the partition base* $part_t(t_i)$ *of an arbitrary tent* $\mathbf{T}$ *with base types* $(t_1, \ldots, t_m)$*. See* [3, 4, 5] *for notations and terminology.*

A significative example with OSB 1.0 has been shown below: this is new in literature and sheds light on important open questions as [8, Problemi 1, 2, 3, pp.5–7] and [9, §3.3].

Example 3.3. Let m=9 and

$$\mathcal{E}=(p_1, b_{\{3,7,9\}}, b_{\{2,6,8\}}, b_{\{1,4,8,9\}}, b_{\{2,6\}}, b_{\{3,7\}}, b_{\{1,5,6,7\}}, b_{\{2,4,8\}}, b_{\{3,4,9\}}).$$

Then $\mathcal{E}^{-1}=(p_1, b_{\{3,5,7,9\}}, b_{\{2,6,7,8\}}, b_{\{7,8,9\}}, b_{\{4,5,6\}}, b_{\{3,7,9\}}, b_{\{2,7,8\}}, b_{\{3,5\}}, b_{\{2,6\}})$.

We have $\mathcal{E} * \mathcal{E}^{-1}=(p_1, \ldots, p_m)$, because $|\mathcal{A}_i| + |\mathcal{C}_i|$ =11 (=$m+2$) for all $i = 1, \ldots, 9$.

Now,
$\mathcal{A}_1$ =(1);(2);(3);(4);(5);(6);(7);(8);(9); $\mathcal{C}_1$ =(1);(2,3,4,5,6,7,8,9,);
$\mathcal{A}_2$ =(2);(6);(8);(1,5,7,);(3,4,9,); $\mathcal{C}_2$ =(2);(1);(9);(3,7,);(4,8,);(5,6,);
$\mathcal{A}_3$ =(3);(7);(9);(1,5,6,);(2,4,8,); $\mathcal{C}_3$ =(3);(1);(8);(2,6,);(4,9,);(5,7,);
$\mathcal{A}_4$ =(4);(1,5,6,7,);(2,8,);(3,9,); $\mathcal{C}_4$ =(4);(1);(5);(8);(9);(3,7,);(2,6,);
$\mathcal{A}_5$ =(5);(1,4,8,9,);(2,6,);(3,7,); $\mathcal{C}_5$ =(5);(1);(4);(6);(7);(3,9,);(2,8,);
$\mathcal{A}_6$ =(6);(2,8,);(1,5,7,);(3,4,9,); $\mathcal{C}_6$ =(6);(1);(2);(5);(9);(3,7,);(4,8,);
$\mathcal{A}_7$ =(7);(3,9,);(1,5,6,);(2,4,8,); $\mathcal{C}_7$ =(7);(1);(3);(5);(8);(2,6,);(4,9,);
$\mathcal{A}_8$ =(8);(2,6,);(1,3,4,5,7,9,); $\mathcal{C}_8$ =(8);(1);(2);(3);(4);(7);(9);(5,6,);
$\mathcal{A}_9$ =(9);(3,7,);(1,2,4,5,6,8,); $\mathcal{C}_9$ =(9);(1);(2);(3);(4);(6);(8);(5,7,).

$D(\mathcal{E})$ is the following:

	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}
t_1	0	1	1	1	1	1	1	1	1	1
t_2	1	1	1	1	1	0	1	0	1	0
t_3	1	1	0	1	1	1	1	1	0	0
t_4	1	1	1	0	1	1	0	1	1	0
t_5	0	0	1	1	0	1	1	1	1	0
t_6	1	1	1	1	0	1	1	0	1	0
t_7	1	0	1	1	1	1	1	1	0	0
t_8	1	1	1	1	1	0	0	1	1	0
t_9	1	1	0	0	1	1	1	1	1	0

$D(\mathcal{E}) = \mathbf{T}$ is decomposable because

$$part_t(t_5) = \{\{1\}, \{5\}, \{2, 3, 4, 6, 7, 8, 9\}\} < p_5.$$

Thus $D(\mathcal{E})$ is a further counterexample for the converse of Proposition 2.3. We notice that, by applying OSB 1.0, we obtain that $\mathcal{E} * \mathcal{E}' < (p_1, \dots, p_m)$. Decomposing $\mathcal{E}$ in exchanges:

$\mathcal{E}_1$=$(p_1, p_9, b_{\{2,6,8\}}, b_{\{2,5,6\}}, b_{\{2,6\}}, b_{\{3,7\}}, b_{\{1,5,6,7\}}, b_{\{2,4,8\}}, b_{\{3,4,9\}})$; $\mathcal{S}_1$=(2,4,{6}).
$\mathcal{E}_2$=$(p_1, p_9, p_5, p_8, b_{\{2,6\}}, b_{\{3,7\}}, b_{\{1,5,6,7\}}, b_{\{2,4,8\}}, b_{\{3,4,9\}})$; $\mathcal{S}_2$=(3,4,{1,2,6,7,8,9}).
$\mathcal{E}_3$=$(p_1, p_9, p_5, p_8, p_2, b_{\{3,7\}}, b_{\{1,5,7\}}, b_{\{2,4,8\}}, b_{\{3,4,9\}})$; $\mathcal{S}_3$=(5,6,{1,2,3,7,8}).
$\mathcal{E}_4$=$(p_1, p_9, p_5, p_8, p_2, p_7, b_{\{2,4,6,8,9\}}, b_{\{2,4,8\}}, b_{\{3,4,9\}})$; $\mathcal{S}_4$=(6,7,{2,4,5,8,9}).
$\mathcal{E}_5$=$(p_1, p_9, p_5, p_8, p_2, p_7, b_{\{2,6,8\}}, b_{\{2,4,8\}}, p_3)$; $\mathcal{S}_5$=(7,9,{2,4,5,8}).
$\mathcal{E}_6$=$(p_1, p_9, p_5, p_8, p_2, p_7, p_6, p_4, p_3)$; $\mathcal{S}_6$=(7,8,{4,5}).

Since $\mathcal{E}_6$ is a permutation, it can be written as product of transpositions as follows

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 9 & 5 & 8 & 2 & 7 & 6 & 4 & 3 \end{pmatrix} = (1)(2, 9, 3, 5)(4, 8)(6, 7)$$

$$= (1)(2,5)(2,3)(2,9)(4,8)(6,7).$$

Consider
$\mathcal{S}_7$=(2,5,{1,3,4,6,7,8,9}), $\mathcal{S}_8$=(2,3,{1,4,5,6,7,8,9}), $\mathcal{S}_9$=(2,9,{1,3,4,5,6,7,8}), $\mathcal{S}_{10}$=(4,8,{1,2,3,5,6,7,9}), $\mathcal{S}_{11}$=(6,7,{1,2,3,4,5,8,9}).
We get $\mathcal{E}_6=\mathcal{S}_7\cdot\mathcal{S}_8\cdot\mathcal{S}_9\cdot\mathcal{S}_{10}\cdot\mathcal{S}_{11}$ and $\mathcal{E}=\mathcal{S}_1\cdot\mathcal{S}_2\cdot\mathcal{S}_3\cdot\mathcal{S}_4\cdot\mathcal{S}_5\cdot\mathcal{S}_6\cdot\mathcal{S}_7\cdot\mathcal{S}_8\cdot\mathcal{S}_9\cdot\mathcal{S}_{10}\cdot\mathcal{S}_{11}$. □

We tested many automorphisms $\mathcal{E}$, by demanding the decomposability of $D(\mathcal{E})$, as in Proposition 2.5, and the conditions $\mathcal{E}*\mathcal{E}^{-1}=\mathcal{E}^{-1}*\mathcal{E}=(p_1,\dots,p_m)$, on the strength of Proposition 2.4. Our tests provided either $\mathcal{E}*\mathcal{E}^{-1}<(p_1,\dots,p_m)$ or $\mathcal{E}^{-1}*\mathcal{E}<(p_1,\dots,p_m)$. Now we show two among many examples.

Example 3.4. Take $m=9$ and assume $|2,3|^{-1}$, $|4,5|^{-1}$, $|6,7|^{-1}$, $|8,9|^{-1}$ belonging to $D(\mathcal{E})$. Construct

$$\mathcal{E}=(b_{\{2,3,4,6,7\}},b_{\{2,3,5\}},b_{\{2,3,6\}},b_{\{2,3,7\}},b_{\{4,5,9\}},b_{\{4,5,8\}},b_{\{1,6,7\}},b_{\{1,3,8,9\}},b_{\{1,2,8,9\}})$$

with

$$\mathcal{E}^{-1}=(b_{\{3,4,7\}},b_{\{1,2,8\}},b_{\{1,2,9\}},b_{\{2,5,6,7\}},b_{\{2,8,9\}},b_{\{3,8,9\}},b_{\{4,8,9\}},b_{\{5,7,8,9\}},b_{\{6,7,8,9\}}).$$

We have
$\mathcal{A}_1$ =(1);(6);(7);(2,3,);(4,5,8,9,); $\mathcal{C}_1$ =(1);(2);(3);(4);(5);(8);(9);(6,7,);
$\mathcal{A}_2$ =(2);(3);(5);(4,6,7,);(1,8,9,); $\mathcal{C}_2$ =(2);(1);(3);(6);(7);(8);(9);(4,5,);
$\mathcal{A}_3$ =(3);(2);(5);(4,6,7,);(1,8,9,); $\mathcal{C}_3$ =(3);(1);(2);(6);(7);(8);(9);(4,5,);
$\mathcal{A}_4$ =(4);(5);(8);(9);(2,3,);(1,6,7,); $\mathcal{C}_4$ =(4);(2);(3);(5);(6);(7);(1,8,9,);
$\mathcal{A}_5$ =(5);(2);(3);(1,8,9,);(4,6,7,); $\mathcal{C}_5$ =(5);(1);(4);(6);(7);(8);(9);(2,3,);
$\mathcal{A}_6$ =(6);(2);(3);(1,8,9,);(4,5,7,); $\mathcal{C}_6$ =(6);(1);(4);(5);(7);(8);(9);(2,3,);
$\mathcal{A}_7$ =(7);(2);(3);(1,8,9,);(4,5,6,); $\mathcal{C}_7$ =(7);(1);(4);(5);(6);(8);(9);(2,3,);
$\mathcal{A}_8$ =(8);(1);(2);(3);(9);(4,5,);(6,7,); $\mathcal{C}_8$ =(8);(4);(5);(6);(7);(2,3,);(1,9,);
$\mathcal{A}_9$ =(9);(1);(2);(3);(8);(4,5,);(6,7,); $\mathcal{C}_9$ =(9);(4);(5);(6);(7);(2,3,);(1,8,);

$D(\mathcal{E}) = \mathbf{T}$ is the following:

	A_1	A_2	A_3	A_4	A_5	A_6	A_7
t_1	1	1	1	1	0	0	0
t_2	0	1	1	1	1	1	1
t_3	0	1	1	1	1	1	1
t_4	1	0	1	1	1	1	1
t_5	1	0	1	1	1	1	1
t_6	1	1	1	0	1	1	1
t_7	1	1	1	0	1	1	1
t_8	1	1	0	1	0	0	1
t_9	1	1	0	1	0	1	0

We have $part_t(t_i) \leq \mathcal{A}_i \; sup \, \mathcal{C}_i < p_i$ for every $i = 1, \ldots, m$ and so $D(\mathcal{E})$ is decomposable, because $\mathcal{A}_i \; sup \; \mathcal{C}_i$ is different from p_i. □

Example 3.5. Let $m = 9$ and assume that $\{3, 5\}^{-1} \notin D(\mathcal{E})$. Give

$$\mathcal{E}{=}(p_1, b_{\{3,5,7\}}, b_{\{2,3,6,7,8\}}, b_{\{7,8,9\}}, b_{\{4,5,6\}}, b_{\{3,7\}}, b_{\{2,3,7,8\}}, b_{\{3,5\}}, b_{\{2,6\}}).$$

and

$$\mathcal{E}^{-1}{=}(p_1, b_{\{3,7,9\}}, b_{\{2,6,8\}}, b_{\{1,4,8,9\}}, b_{\{2,6\}}, b_{\{3,7\}}, b_{\{2,8\}}, b_{\{3,6,9\}}, b_{\{1,5,7\}}).$$

Now,
$\mathcal{A}_1$ =(1);(2);(3);(4);(5);(6);(7);(8);(9); $\mathcal{C}_1$ =(1);(2,3,4,5,6,7,8,9,);
$\mathcal{A}_2$ =(2);(6);(3,7,8,);(1,4,5,9,); $\mathcal{C}_2$ =(2);(1);(3);(5);(7);(8,9,);(4,6,);
$\mathcal{A}_3$ =(3);(5);(7);(1,2,4,6,8,9,); $\mathcal{C}_3$ =(3);(1);(2);(6);(9);(7,8,);(4,5,);
$\mathcal{A}_4$ =(4);(1);(7,8,9,);(3,5,);(2,6,); $\mathcal{C}_4$ =(4);(5);(6);(3,7,);(2,8,);(1,9,);
$\mathcal{A}_5$ =(5);(3,7,);(1,2,4,6,8,9,); $\mathcal{C}_5$ =(5);(1);(2);(3);(4);(6);(9);(7,8,);
$\mathcal{A}_6$ =(6);(2,3,7,8,);(1,4,5,9,); $\mathcal{C}_6$ =(6);(1);(2);(3);(4);(5);(7);(8,9,);
$\mathcal{A}_7$ =(7);(3,5,);(1,2,4,6,8,9,); $\mathcal{C}_7$ =(7);(1);(2);(3);(6);(8);(9);(4,5,);
$\mathcal{A}_8$ =(8);(3,7,);(2,6,);(1,4,5,9,); $\mathcal{C}_8$ =(8);(1);(2);(3);(5);(7);(9);(4,6,);
$\mathcal{A}_9$ =(9);(1);(4,5,6,);(2,3,7,8,); $\mathcal{C}_9$ =(9);(3);(5);(7);(8);(2,6,);(1,4,);

$D(\mathcal{E}) = \mathbf{T}$ is the following:

	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}	A_{11}	A_{12}	A_{13}
t_1	1	1	0	0	0	1	1	0	1	1	1	1	1
t_2	1	1	1	1	1	1	1	0	0	0	0	1	1
t_3	0	1	1	1	1	1	1	1	1	1	1	1	1
t_4	1	1	0	0	1	0	0	0	0	1	1	0	1
t_5	1	1	1	1	1	0	1	1	1	1	1	1	1
t_6	1	1	1	1	1	1	1	0	0	0	1	0	1
t_7	0	0	1	1	1	1	1	1	1	1	1	1	1
t_8	1	0	1	1	1	1	1	0	0	1	0	1	0
t_9	1	1	0	1	0	1	0	0	0	1	1	1	0

Note that also here $\mathcal{A}_8 \; sup \, \mathcal{C}_8$ is different from p_8 so that $D(\mathcal{E})$ is decomposable, because $part_t(t_8) \leq \mathcal{A}_8 \; sup \; \mathcal{C}_8 < p_8$. Note that $part_t(t_8) < p_8$.

Since $\{3,5\}^{-1} \notin D(\mathcal{E})$, we have

$$part_t(t_4) < p_4 = \mathcal{A}_4 \ sup \ \mathcal{C}_4$$

and

$$part_t(t_7) < p_7 = \mathcal{A}_7 \ sup \ \mathcal{C}_7.$$

□

We used OSB 1.0 for constructing an example in which $D(\mathcal{E})$ is decomposable and $\mathcal{E} * \mathcal{E}^{-1} = \mathcal{E}^{-1} * \mathcal{E} = (p_1, \dots, p_m)$. All our tests show that this doesn't happen. Therefore, the following conjecture, which can be found also in [9, §3.3], seems to be true.

Conjecture. *If $\mathcal{E} * \mathcal{E}^{-1} = \mathcal{E}^{-1} * \mathcal{E} = (p_1, \dots, p_m)$, then $D(\mathcal{E})$ is indecomposable.*

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