PROBABILISTIC PROPERTIES OF THE RELATIVE TENSOR DEGREE OF FINITE GROUPS

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ABSTRACT. Denoting by $H \otimes K$ the nonabelian tensor product of two subgroups H and K of a finite group G, we investigate the relative tensor degree $d^{\otimes}(H,K) = \frac{|\{(h,k) \in H \times K \ | \ h \otimes k = 1\}|}{|H| \ |K|}$ of H and K. The case $H = K = G$ has been studied recently. Here we deal with arbitrary subgroups H and K, showing analogies and differences between $d^{\otimes}(H,K)$ and the relative commutativity degree $d(H,K) = \frac{|\{(h,k) \in H \times K \ | \ [h,k]=1\}|}{|H| \ |K|}$, which is a generalization of the probability of commuting elements, introduced by Erdős.

1. Brown's terminology for nonabelian tensor products

We will consider finite group only. Brown and others studied the nonabelian tensor products of groups in two classical works [2, 3] almost thirty years ago. There has been a wide production in algebra and topology after these fundamental papers, because they have shown interesting relations between various areas of pure mathematics. In the context of the nonabelian tensor products, the well known notion of *Schur multiplier of a group* may be generalized to that of *Schur multiplier of a triple of groups*.

Following [4, Section 6], a *triple* (G,H,K) is a group G with two normal subgroups H and K and the Schur multiplier of (G,H,K) is an abelian group denoted by $M(G,H,K)$ and defined in terms of the *mapping cone* $B(G,H,K)$ of the *canonical cofibration* $B(G,K) \rightarrow B(G/K, HK/H)$. These notions involve some homological algebra and are defined via exact sequences in [4, (22) and (23), p. 368]. We refer in fact to [4] for feedback on mapping cone, canonical cofibration and Schur multiplier of a triple of a group.

From [2, 3], a group G acts by conjugation on its normal subgroups H and K via the rule ${}^{g}x = gxg^{-1}$, for g in G and x in H (or K), and the *nonabelian tensor product* $H \otimes K$ is defined as the group generated by the symbols $h \otimes k$, subject to the relations:

$$h_1h_2 \otimes k_1 = ({}^{h_1}h_2 \ \otimes \ {}^{h_1}k_1) \ (h_1 \otimes k_1) \text{ and } k_1k_2 \otimes h_1 = (k_1 \otimes h_1) \ ({}^{k_1}h_1 \ \otimes \ {}^{k_1}k_2),$$

where $h_1, h_2 \in H$ and $k_1, k_2 \in K$. Adding the relation $a \otimes a = 1$ for all $a \in H \cap K$, we have the *nonabelian exterior product* $H \wedge K$ of H and K. This can be seen equivalently by introducing the *diagonal subgroup*

$$\nabla(H \cap K) = \langle a \otimes a \mid a \in H \cap K \rangle$$

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and noting that

$$H \otimes K/\nabla(H \cap K) = H \wedge K.$$

In particular, we denote $\nabla(H\cap K)$ by $\nabla(G)$, when $H = K = G$. On another hand, the map

$$\kappa' : h \wedge k \in H \wedge K \mapsto \kappa'(h \wedge k) = [h, k] = hkh^{-1}k^{-1} \in [H, K]$$

is an epimorphism of groups such that

$$\ker \kappa' \simeq M(G, H, K),$$

whenever $G = HK$ and this is a very useful way to look at $M(G, H, K)$ (see [4, Theorem 6.1]). Even the map

$$\kappa : h \otimes k \in H \otimes K \mapsto \kappa(h \otimes k) = [h, k] \in [H, K]$$

is an epimorphism of groups such that $\ker \kappa$ is an abelian group, denoted by $J(G, H, K)$. If $G = HK$ (with H and K normal in G), then the following diagram is commutative

$$(*) \qquad \begin{array}{ccccccccc} 1 & \longrightarrow & J(G,H,K) & \longrightarrow & H \otimes K & \xrightarrow{\kappa} & [H,K] & \longrightarrow & 1 \\ & & \pi\big\downarrow & & \varepsilon\big\downarrow & & \big\| & & \\ 1 & \longrightarrow & M(G,H,K) & \longrightarrow & H \wedge K & \xrightarrow{\kappa'} & [H,K] & \longrightarrow & 1 \end{array}$$

with central extensions as rows and natural epimorphisms

$$\pi : h \otimes k \in J(G, H, K) \mapsto h \wedge k \in M(G, H, K),$$

$$\varepsilon : h \otimes k \in H \otimes K \mapsto h \wedge k \in H \wedge K$$

as columns. Of course, if $G = H = K$, we have that $M(G) = H_2(G, \mathbb{Z})$ is exactly the *Schur multiplier* of G (i.e.: the second group of homology with integral coefficients on G), $G \otimes G$ is the *nonabelian tensor square* of G and $G \wedge G$ is the *nonabelian exterior square* of G. Some properties of these interesting structures are discussed in [14] in terms of generators and relations, but we recall some specific results of computational nature later on.

The idea of the present paper is to use the nonabelian tensor product, in order to define a notion of probability, which involves the number of commuting pairs with respect to the operator $\otimes$, introduced above. This will be formalized in Section 3 and we will see that it is related to the concept of probability of commuting elements, largely studied in [5, 7, 8, 9]. Several families of metacyclic groups play an important role in these investigations so that Section 2 is devoted to specialize most of the recent results in literature for dihedral groups and generalized quaternion groups. These results will be used later on for examples and counterexamples to the general theorems of probabilistic nature. The main results of the present paper are placed in fact in Section 5, after some technical preliminaries in Section 4.

2. Some computations on dihedral and generalized quaternion groups

The present section specializes some results in [2] to our context. Writing

$$(\dagger) \qquad D_{2n} = C_2 \ltimes C_n = \langle a, b \mid b^2 = a^n = 1, b^{-1}ab = a^{-1}\rangle$$

for the dihedral group of order $2n$ (where $n \geq 1$ is arbitrary) and

$$(\dagger\dagger) \qquad Q_{2^n} = \langle a, b \mid b^4 = a^{2^{n-1}} = 1, b^2 = a^{2^{n-2}}, b^{-1}ab = a^{-1}\rangle$$

for the generalized quaternion 2–group of order 2^n, we recall [2, Propositions 12, 13 and 14].

Lemma 2.1. *Assume that Q_{2^n} is presented by (††), D_{2n} by (†) and C_t is cyclic of order $t \geq 2$. Then*

(i) *for all $n \geq 4$ we have $Q_{2^n} \otimes Q_{2^n} = C_2 \times C_{2^{n-1}} \times C_2 \times C_2$, where the first factor is generated by $(b\otimes b)(b \otimes a)^{2^{n-3}}(a\otimes a)$, the second by $b\otimes a$, the third by $a \otimes a$ and the fourth by $(b \otimes a)(a \otimes b)$.*

(ii) *for $n = 3$ we have $Q_8 \otimes Q_8 = C_2 \times C_4 \times C_4 \times C_2$, where the first factor is generated by $(b \otimes b)(b \otimes a)(a \otimes a)$, the second by $b \otimes a$, the third by $a \otimes a$ and the fourth by $(b \otimes a)(a \otimes b)$.*

(iii) *for all odd $n \geq 1$ we have $D_{2n} \otimes D_{2n} = C_2 \times C_n$, where the first factor is generated by $b \otimes b$ and the second by $b \otimes a$; for all even $n \geq 1$ we have $D_{2n}\otimes D_{2n} = C_2\times C_n \times C_2 \times C_2$, where the first factor is generated by $b\otimes b$, the second by $b \otimes a$, the third by $a \otimes a$ and the fourth by $(b \otimes a)(a \otimes b)$.*

Proof. (i). This is exactly [2, Proposition 13, p. 191]. (ii). This follows again from [2, Proposition 13, p. 191]. (iii). This follows from [2, Proposition 14, p. 191]. □

Now we can do further computations, involving the homological functors in (∗).

Lemma 2.2. *Let (††) be a presentation for Q_{2^n} and a the generator of Q_{2^n} of order 2^{n-1} in (††). Then*

(i) *$J(Q_{2^n}, Q_{2^n}, \langle a\rangle) = C_2 \times C_2$ for all $n \geq 4$;*

(ii) *$M(Q_{2^n}, Q_{2^n}, \langle a\rangle) = C_2$ for all $n \geq 4$;*

(iii) *$J(Q_8, Q_8, \langle a\rangle) = C_4 \times C_2$;*

(iv) *$M(Q_8, Q_8, \langle a\rangle) = C_2$.*

Proof. (i). In order to prove our claim, it is enough to consider κ in (∗), specialized to our context. We begin to compute $Q_{2^n} \otimes \langle a\rangle$, where of course $\langle a\rangle = C_{2^{n-1}}$. Lemma 2.1 (i) allows us to describe the generators of $Q_{2^n} \otimes \langle a\rangle$: there are no elements of the form $a \otimes b$ and $b \otimes b$. Then $Q_{2^n} \otimes \langle a\rangle$ contains only elements of the form $b \otimes a$ and $a \otimes a$ (or products of these). On the other hand, $Q_{2^n} \otimes \langle a\rangle$ must be a subgroup of $Q_{2^n} \otimes Q_{2^n}$, which is completely described by Lemma 2.1 (i) in terms of $C_2 \times C_{2^{n-1}} \times C_2 \times C_2$, and $Q_{2^n} \otimes \langle a\rangle$ contains the generators of the second and third factors of this product, while it cannot contain the aforementioned elements. Then we conclude $Q_{2^n} \otimes \langle a\rangle = C_{2^{n-1}} \times C_2$. On the other hand, the relations of (††) imply $[a, a] = 1$ and $[b, a] = a^2$, where a^2 has order 2^{n-2}, so $[Q_{2^n}, \langle a\rangle] = \langle a^2\rangle = C_{2^{n-2}}$. Now $\kappa : Q_{2^n} \otimes \langle a\rangle \to [Q_{2^n}, \langle a\rangle]$ is completely determined by its values on the generators, that is, by $\kappa(a \otimes a) = 1$ and $\kappa(b \otimes a) = [b, a] = a^2$. Note that $\kappa((b \otimes a)^2) = \kappa(b \otimes a)^2 = a^4$ and so on, getting to $\kappa((b \otimes a)^{2^{n-2}}) = \kappa(b \otimes a)^{2^{n-2}} = a^{2^{n-1}} = 1$. This implies $a\otimes a \in \ker\kappa$ and $(b\otimes a)^{2^{n-2}} \in \ker\kappa$ and any other element of $\ker\kappa$ is generated by these two, so that $\ker\kappa = J(Q_{2^n}, Q_{2^n}, \langle a\rangle) = \langle(b \otimes a)^{2^{n-2}}\rangle \times \langle a \otimes a\rangle = C_2 \times C_2$.

(ii). Applying again Lemma 2.1 (i) and repeating the previous argument of (i), we find $Q_{2^n} \wedge \langle a\rangle = C_{2^{n-1}}$, since the generator $a\otimes a$ must be omitted by definition this time. Here $\kappa' : Q_{2^n} \wedge \langle a\rangle \to [Q_{2^n}, \langle a\rangle]$ is completely determined by its values on the generators a and b. Since (∗) is a central extension, $(Q_{2^n} \wedge \langle a\rangle)/M(Q_{2^n}, Q_{2^n}, \langle a\rangle) \simeq [Q_{2^n}, \langle a\rangle]$, but $[Q_{2^n}, \langle a\rangle] = C_{2^{n-2}}$ and $Q_{2^n}\wedge\langle a\rangle = C_{2^{n-1}}$ in an abelian context, hence $\ker\kappa' \simeq M(Q_{2^n}, Q_{2^n}, \langle a\rangle) = C_2$.

(iii). We may repeat the argument of (i) above, using Lemma 2.1 (ii) instead of Lemma 2.1 (i). Consequently, Q_{2^n} must be replaced by Q_8, but here $Q_{2^n} \otimes \langle a \rangle = C_2 \times C_2$ becomes $Q_8 \otimes \langle a \rangle = C_4 \times C_4$. The rest follows mutatis mutandis.

(iv). We may apply (iii) and the argument of (ii) above. □

As noted in [2, p. 191], we may consider a dihedral group as a suitable central quotient of a generalized quaternion group and so the following computations specialize Lemma 2.2.

Lemma 2.3. *Using the presentation* (†), *we have*

(i) $J(D_{2n}, D_{2n}, \langle a \rangle) = 1$, *if n is odd;*
(ii) $J(D_{2n}, D_{2n}, \langle a \rangle) = C_2 \times C_2$, *if n is even;*
(iii) $M(D_{2n}, D_{2n}, \langle a \rangle) = 1$, *if n is odd;*
(iv) $M(D_{2n}, D_{2n}, \langle a \rangle) = C_2$, *if n is even.*

Proof. (i). We want to apply the argument of Lemma 2.2 (i), using Lemma 2.1 (iii). Of course, we need to compute $D_{2n} \otimes \langle a \rangle$, where $\langle a \rangle = C_n$. Lemma 2.1 (iii) shows that the generators of $D_{2n} \otimes \langle a \rangle$ are of the form $b \otimes b$ for the factor C_2 and $b \otimes a$ for the factor C_n. Since we haven't generators for C_2, we get $D_{2n} \otimes \langle a \rangle = \langle b \otimes a \rangle = C_n$. On the other hand, the relations of (†) imply $C_{D_{2n}}(a) = \langle a \rangle$, so $[D_{2n}, \langle a \rangle] = \langle a \rangle$. Now $\kappa : D_{2n} \otimes \langle a \rangle \to [D_{2n}, \langle a \rangle]$ is an epimorphism completely determined by its values on the generators. The exactness of $(*)$ implies $(D_{2n} \otimes \langle a \rangle)/\ker \kappa \simeq [D_{2n}, \langle a \rangle] = \langle a \rangle$ and so $\ker \kappa = J(D_{2n}, D_{2n}, \langle a \rangle) = 1$.

(ii). We may overlap (i) above, using Lemma 2.1 (iii) when n is even, but the involutions need to be treated carefully. Here $D_{2n} \otimes \langle a \rangle = \langle b \otimes a \rangle \times \langle a \otimes a \rangle = C_n \times C_2$, but now $[D_{2n}, \langle a \rangle] = \langle a^2 \rangle = C_{n/2}$. Since $\kappa : D_{2n} \otimes \langle a \rangle \to [D_{2n}, \langle a \rangle]$ is completely determined by its values on the generators, the conditions $\kappa(a \otimes a) = 1$ and $\kappa((b \otimes a)^{n/2})) = 1$ imply $J(D_{2n}, D_{2n}, \langle a \rangle) = \langle (b \otimes a)^{n/2} \rangle \times \langle a \otimes a \rangle = C_2 \times C_2$.

(iii). If n is odd, Lemma 2.1 (iii) shows that $D_{2n} \otimes \langle a \rangle = D_{2n} \wedge \langle a \rangle$. Then we should replace the role of κ with that of κ' and the proof follows.

(iv). If n is even, (ii) above and Lemma 2.1 (iii) show that $D_{2n} \otimes \langle a \rangle / \langle a \otimes a \rangle = D_{2n} \wedge \langle a \rangle = C_n$. As in (ii) above, we find $[D_{2n}, \langle a \rangle] = \langle a^2 \rangle = C_{n/2}$ and $\kappa' : D_{2n} \wedge \langle a \rangle \to [D_{2n}, \langle a \rangle]$ is completely determined by its values on the generators. The condition $\kappa((b \wedge a)^{n/2}) = 1$ implies $M(D_{2n}, D_{2n}, \langle a \rangle) \simeq \ker \kappa' = \langle (b \wedge a)^{n/2} \rangle = C_2$. □

3. Commutativity, exterior and tensor degrees

Assigned a group G with two normal subgroups H and K, we may form $H \otimes K$ and $H \wedge K$, defining the *tensor centralizer*

$$C^{\otimes}_K(H) = \{k \in K \mid h \otimes k = 1, \forall h \in H\} = \bigcap_{h \in H} \{k \in K \mid h \otimes k = 1\} = \bigcap_{h \in H} C^{\otimes}_K(h)$$

of H in K, and the *exterior centralizer*

$$C^{\wedge}_K(H) = \{k \in K \mid h \wedge k = 1, \forall h \in H\} = \bigcap_{h \in H} \{k \in K \mid h \wedge k = 1\} = \bigcap_{h \in H} C^{\wedge}_K(h)$$

of H in K. The set $C^{\otimes}_K(H)$ is actually a subgroup of K (see [1, 12, 13]) and, in particular, $C^{\otimes}_G(G) = Z^{\otimes}(G)$ is a subgroup of G, called *tensor center* of G. Moreover, one can see that $C^{\otimes}_K(H) \subseteq C_K(H)$ and $Z^{\otimes}(G) \subseteq Z(G)$. Analogously, this happens for $C^{\wedge}_K(H)$ and, in particular, $C^{\wedge}_G(G) = Z^{\wedge}(G)$ is called *exterior center* of G (see [10, 11, 13]).

Taking as model the theory of the probability of commuting pairs of elements (see [5, 6, 7, 8, 9, 16] but also [15] for recent generalizations), the *relative commutativity degree* is introduced in [5] as the ratio

$$d(H,K) = \frac{|\{(h,k) \in H \times K \mid [h,k] = 1\}|}{|H|\ |K|} = \frac{1}{|H|\ |K|} \sum_{h \in H} |C_K(h)| = \frac{k_K(H)}{|H|},$$

where H and K are two arbitrary subgroups of G, not necessarily normal in G, and $k_K(H)$ is the number of $K-$conjugacy classes that constitute H. When $G = H = K$, $d(G) = d(G,G) = k_G(G)/G = k(G)/G$ is the well known *commutativity degree* of G. On another hand, it is studied already in [11, 13] the *relative exterior degree*

$$d^{\wedge}(H,K) = \frac{|\{(h,k) \in H \times K \mid h \wedge k = 1\}|}{|H|\ |K|} = \frac{1}{|H|\ |K|} \sum_{h \in H} |C^{\wedge}_K(h)|,$$

of H and K (but here H and K must be normal in G in order to form $H \wedge K$). The case $G = H = K$, that is, $d^{\wedge}(G,G) = d^{\wedge}(G)$, was originally called *exterior degree* of G. It is easy to see that $d^{\wedge}(G) = 1$ if and only if $Z^{\wedge}(G) = G$, that is, if and only if the exterior center equals the whole group.

We recall that a group G is *capable*, if $G \simeq E/Z(E)$ for some group E. Moreover, one can see that G is capable if and only if $Z^{\wedge}(G) = 1$. This means that the relative exterior degree turns out to be very useful. It does not represent only the probability that two randomly chosen elements commute with respect to the operator $\wedge$, but it allows us to decide how far is the group from being capable.

Finally, the papers [1, 12] are devoted to the *relative tensor degree*

$$d^{\otimes}(H,K) = \frac{|\{(h,k) \in H \times K \mid h \otimes k = 1\}|}{|H|\ |K|} = \frac{1}{|H|\ |K|} \sum_{h \in H} |C^{\otimes}_K(h)|$$

of the normal subgroups H and K of G. While [12] describes the *tensor degree* $d^{\otimes}(G) = d^{\otimes}(G,G)$, which is equal to 1 if and only if $Z^{\otimes}(G) = G$, the paper [1] investigates some qualitative relations between $d^{\otimes}(H,K)$, $d^{\wedge}(H,K)$ and $d(H,K)$. Looking at these two works, we may find restrictions on the tensor center of G by its tensor degree and deduce information on the size of $J(G) = J(G,G,G)$, which has an important interpretation in algebraic topology. Unfortunately, few results of this kind are available at the moment and these are contained mainly in [1, 12]. We will see that $d^{\otimes}(H,K)$ has some properties of measure theory, which were known already for $d(H,K)$ in [5, 6, 7, 8], and we will test them mostly on dihedral and generalized quaternion groups.

4. Some previous results

We recall three results from [1] in the following lines without proofs. The first is a technical lemma, which helps in many computations.

Lemma 4.1 (See [1], Lemma 2.1)**.** *Let H and K be two normal subgroups of a group G. Then*

$$d^{\otimes}(H,K) = \frac{1}{|H|} \sum_{i=1}^{k_K(H)} \frac{|C^{\otimes}_K(h_i)|}{|C_K(h_i)|},$$

where $h_i \in H$ are representatives of the K-conjugacy classes in H. In particular, if $G = HK$, then $C_K(h_i)/C^{\otimes}_K(h_i)$ is isomorphic to a subgroup of $J(G,H,K)$ and $|C_K(h_i) : C^{\otimes}_K(h_i)| \leq |J(G,H,K)|$ for all $i = 1, \ldots, k_K(H)$.

The second is a fundamental relation among the relative commutativity degree, the relative tensor degree and the relative exterior degree.

Theorem 4.2 (See [1], Theorem 1.2)**.** *Let H, K be two normal subgroups of a group G. Then*

$$d^{\otimes}(H,K) \leq d^{\wedge}(H,K) \leq d(H,K).$$

Moreover, if $J(G,H,K)$ is trivial, then $d^{\otimes}(H,K) = d^{\wedge}(H,K) = d(H,K)$.

Incidentally, Theorem 4.2 may be combined with [9, Theorem 1], in order to find interesting conditions of solvability. More precisely, assume that a group G is neither isoclinic to A_4 nor such that its central quotient $G/Z(G)$ is isoclinic to A_4. Then G is supersolvable, provided $H = K = G$ and $d^{\otimes}(G) > 5/16$. The third is a variation of important bounds in [5, 6, 7, 8, 11].

Theorem 4.3 (See [1], Theorem 1.1)**.** *Let H, K be two normal subgroups of a group G. Then*

$$\frac{d(H,K)}{|J(G,H,K)|} \leq d^{\otimes}(H,K) \leq d(H,K).$$

In particular, if $J(G,H,K)$ is trivial, then $d^{\otimes}(H,K) = d(H,K)$.

A condition of equality in the lower bound of Theorem 4.2 is the following.

Corollary 4.4. *Let H and K be two normal subgroups of a group G such that $H \cap K = 1$. Then $d^{\otimes}(H,K) = d^{\wedge}(H,K)$.*

Proof. Since $H \wedge K = H \otimes K/\nabla(H \cap K)$, we conclude that $H \wedge K = H \otimes K$, whenever $H \cap K = 1$. □

On another hand, there is a numerical consequence of Theorem 4.3, which deals with a specific value of relative tensor degree.

Corollary 4.5. *Let H and K be two normal subgroups of a group G and p the smallest prime divisor of $|G|$. If $[H,K] \neq 1$, then $d^{\otimes}(H,K) \leq \frac{2p-1}{p^2}$. In particular, $d^{\otimes}(H,K) \leq 3/4$.*

Proof. From [11, Proposition 2.10] we recall that $d^{\wedge}(H,K) \leq \frac{2p-1}{p^2}$. Now Theorem 4.2 implies $d^{\otimes}(H,K) \leq d^{\wedge}(H,K) \leq \frac{2p-1}{p^2}$ and, in particular, $\frac{2p-1}{p^2} \leq \frac{3}{4}$. □

Further useful information can be found in [12], once we focus on dihedral and quaternion groups. We have in fact:

Corollary 4.6 (See [12], Corollary 4.4)**.** *The following statements are true:*

(i) *$d^{\otimes}(Q_{2^n}) = \frac{2^{n-3}+2^{n-4}+1}{2^n}$ and $\frac{2^{n-2}+3}{2^n} = d^{\wedge}(Q_{2^n}) = d(Q_{2^n})$ for all $n > 3$. Moreover, $d^{\otimes}(Q_8) = \frac{1}{4}$ and $\frac{5}{8} = d^{\wedge}(Q_8) = d(Q_8)$ in case $n = 3$. In particular, $d^{\otimes}(Q_{2^n}) < d^{\wedge}(Q_{2^n}) = d(Q_{2^n})$ for all $n \geq 3$.*

(ii) *$d^{\otimes}(D_{2^n}) = \frac{2^{n-3}+2^{n-4}+1}{2^n}$ and $d^{\wedge}(D_{2^n}) = d(D_{2^n}) = \frac{2^{n-2}+3}{2^n}$ for all $n > 3$. Moreover, $d^{\otimes}(D_8) = \frac{5}{16}$ and $d^{\wedge}(D_8) = d(D_8) = \frac{5}{8}$ in case $n = 3$. In particular, $d^{\otimes}(D_{2^n}) < d^{\wedge}(D_{2^n}) = d(D_{2^n})$ for all $n \geq 3$.*

(iii) $d^{\otimes}(C_p^{(n)}) = \frac{2p^n-1}{p^{2n}}$; $d^{\wedge}(C_p^{(n)}) = \frac{p^n+p^{n-1}-1}{p^{2n-1}}$; $d(C_p^{(n)}) = 1$. *In particular,* $d^{\otimes}(C_p^{(n)}) < d^{\wedge}(C_p^{(n)}) < d(C_p^{(n)})$ *for all* $n \geq 1$ *and primes* $p \geq 2$.

A special case is recalled separately.

Example 4.7. Given the presentation (††), we claim that $d^{\otimes}(Q_8, \langle a \rangle) = 3/8$. A first argument comes from a direct computations of the tensor centralizers, looking at Lemmas 2.1, 2.2 and at the tensor centralizers of Q_8, described in [12]. A second way is of computational nature, using [17]. This is described below.

```
LoadPackage("HAP");
f:=FreeGroup(2);
x:=f.1;y:=f.2;
g:=f/[x^4,x^2*y^(-2),y^(-1)*x*y*x];
p:=Subgroups(g);;
n:=p[3];
m:=Elements(n);;
e:=Elements(g);;
l:=NonabelianTensorProduct(g,n);;
t:=0;;
for j in [1..Size(n)] do
c:=[];
for i in [1..Size(g)] do
if l.pairing(e[i],m[j])=l.pairing(e[1],m[1]) then Add(c,e[i]);fi;od;
t:=t+Size(c); od;
Print("d⊗(G,N)=",t/(Size(g)*Size(n))," \n ");
```

The following examples improve computations in Corollary 4.6 (i) and (ii).

Example 4.8. Let n be an odd integer. With the presentation (†) for D_{2n}, we know that $d^{\wedge}(D_{2n}, \langle a \rangle) = (n+1)/2n$ from [13, Example 4.1] and that $d(D_{2n}, \langle a \rangle) = (n+1)/2n$ from [5, Example 3.11]. In other words, the relative exterior and the relative commutativity degree of D_{2n} and $\langle a \rangle$ coincide. The same value can be found for the relative tensor degree of D_{2n} and $\langle a \rangle$, because

$$d^{\otimes}(D_{2n}, \langle a \rangle) = \frac{1}{|D_{2n}|\ |\langle a \rangle|} \sum_{x \in \langle a \rangle} |C^{\otimes}_{D_{2n}}(x)|$$

$$= \frac{1}{2n^2}(|C^{\otimes}_{D_{2n}}(1)| + |C^{\otimes}_{D_{2n}}(a)| + \ldots + |C^{\otimes}_{D_{2n}}(a^{n-1})|) = \frac{2n + (n-1)n}{2n^2} = \frac{n+1}{2n}.$$

On the other hand, Lemma 2.3 (i) shows that $J(D_{2n}, D_{2n}, \langle a \rangle)$ is trivial and so Theorem 4.2 applies.

Now let n be an even integer. Again from [13, Example 4.1], we can see that the above value of relative exterior degree does not change. Similarly, this happens for the relative commutativity degree by [5, Example 3.11]. But now

$$d^{\otimes}(D_{2n}, \langle a \rangle) = \frac{1}{|D_{2n}|\ |\langle a \rangle|} \sum_{x \in \langle a \rangle} |C^{\otimes}_{D_{2n}}(x)|$$

$$= \frac{1}{2n^2}(|C^{\otimes}_{D_{2n}}(1)| + |C^{\otimes}_{D_{2n}}(a)| + \ldots + |C^{\otimes}_{D_{2n}}(a^{n-1})|)$$

$$= \frac{1}{2n^2}(|C^{\otimes}_{D_{2n}}(1)| + \underbrace{|C^{\otimes}_{D_{2n}}(a^2)| + |C^{\otimes}_{D_{2n}}(a^4)| + |C^{\otimes}_{D_{2n}}(a^6)| + \ldots}_{(\frac{n}{2}-1)-\text{times}}$$

$$+ \underbrace{|C^{\otimes}_{D_{2n}}(a)| + |C^{\otimes}_{D_{2n}}(a^3)| + |C^{\otimes}_{D_{2n}}(a^5)| + \ldots}_{\frac{n}{2}-\text{times}})$$

$$= \frac{1}{2n^2}\left(2n + \left(\frac{n}{2} - 1\right)n + \frac{n}{2}\cdot\frac{n}{2}\right) = \frac{3n+4}{8n},$$

since we may overlap the computations in [12, Theorem 4.3] finding $C^{\otimes}_{D_{2n}}(a^{2k}) = \langle a\rangle$ and $C^{\otimes}_{D_{2n}}(a^{2k+1}) = \langle a^2\rangle$. Here the situation is very interesting: we have

$$C^{\wedge}_{D_{2n}}(a) = 1 \quad \text{and} \quad d^{\wedge}(D_{2n}, \langle a\rangle) = d(D_{2n}, \langle a\rangle),$$

according to [13, Example 4.1]. Here $J(D_{2n}, D_{2n}, \langle a\rangle)$ and $M(D_{2n}, D_{2n}, \langle a\rangle)$ are computed in Lemma 2.3 (ii) and (iv). Theorem 4.2 does not apply, but Theorem 4.3 is satisfied and in fact

$$\frac{d(D_{2n}, \langle a\rangle)}{|J(D_{2n}, D_{2n}, \langle a\rangle)|} = \frac{n+1}{8n} < \frac{3n+4}{8n} = d^{\otimes}(D_{2n}, \langle a\rangle) < \frac{n+1}{2n} = d(D_{2n}, \langle a\rangle).$$

Therefore we have examples of families of groups for which the notion of relative tensor degree is significant.

The presence of rich examples (and of large families of groups) for which tensor, exterior and commutativity degrees are different testifies the relevance of the probabilistic methods in the study of the structural properties of groups. To this scope, we end with an analysis of the generalized quaternion groups.

Example 4.9. Let $n \geq 4$ be an integer. With the above presentation for Q_{2^n}, we know that Q_{2^n} possesses a cyclic maximal normal subgroup $\langle a\rangle \simeq C_{2^{n-1}}$. From [12, Theorem 4.3] it is possible to note that $C^{\otimes}_{Q_{2^n}}(a^{2k}) = \langle a\rangle$ and $C^{\otimes}_{Q_{2^n}}(a^{2k+1}) = \langle a^2\rangle$ and an argument of Example 4.8 may be adapted. We have in fact

$$d^{\otimes}(Q_{2^n}, \langle a\rangle) = \frac{1}{|Q_{2^n}|\ |\langle a\rangle|}\sum_{x\in\langle a\rangle}|C^{\otimes}_{Q_{2^n}}(x)|$$

$$= \frac{1}{2^n\cdot 2^{n-1}}(|C^{\otimes}_{Q_{2^n}}(1)| + |C^{\otimes}_{Q_{2^n}}(a)| + \ldots + |C^{\otimes}_{Q_{2^n}}(a^{2^{n-1}})|)$$

$$= \frac{1}{2^n\cdot 2^{n-1}}(|C^{\otimes}_{Q_{2^n}}(1)| + \underbrace{|C^{\otimes}_{Q_{2^n}}(a^2)| + |C^{\otimes}_{Q_{2^n}}(a^4)| + |C^{\otimes}_{Q_{2^n}}(a^6)| + \ldots}_{(2^{n-2}-1)-\text{times}}$$

$$+ \underbrace{|C^{\otimes}_{Q_{2^n}}(a)| + |C^{\otimes}_{Q_{2^n}}(a^3)| + |C^{\otimes}_{Q_{2^n}}(a^5)| + \ldots}_{2^{n-2}-\text{times}})$$

$$= \frac{1}{2^n\cdot 2^{n-1}}(2^n + 2^{n-2}\cdot 2^{n-2} + (2^{n-2}-1)2^{n-1}) = \frac{3\cdot 2^{n-3}+1}{2^n}.$$

Here $J(Q_{2^n}, Q_{2^n}, \langle a\rangle)$, $M(Q_{2^n}, Q_{2n}, \langle a\rangle)$, $J(Q_8, Q_8, \langle a\rangle)$, $M(Q_8, Q_8, \langle a\rangle)$ are nontrivial and computed in Lemma 2.2. This indicates that commutativity, tensor and exterior degree are different and very interesting to compare. For instance, when $n = 3$, we have seen by Example 4.7 that $d^{\otimes}(Q_8, \langle a\rangle) = 3/8$. Incidentally, this value is reached asymptotically in the following way:

$$d^{\otimes}(Q_8, \langle a\rangle) = \lim_{n>3} d^{\otimes}(Q_{2^n}, \langle a\rangle) = \lim_{n\geq 1} d^{\otimes}(D_{2^n}, \langle a\rangle) = \frac{3}{8}.$$

Finally, we may look at [5, Example 3.11 (ii)] and [13, Example 4.2], and we find that for all $n \geq 4$

$$d^{\otimes}(Q_{2^n}, \langle a\rangle) = \frac{3\cdot 2^{n-3}+1}{2^n} < \frac{2^{n-1}+1}{2^n} = d^{\wedge}(Q_{2^n}, \langle a\rangle) < \frac{2^{n-1}+2}{2^n} = d(Q_{2^n}, \langle a\rangle)$$

and for $n = 3$

$$\frac{3}{8} = d^{\otimes}(Q_8, \langle a\rangle) < \frac{5}{8} = d^{\wedge}(Q_8, \langle a\rangle) < \frac{3}{4} = d(Q_8, \langle a\rangle).$$

5. Main theorems

An improvement of the upper bound in Theorem 4.3 is offered by the next result.

Theorem 5.1. *Let H, K be two normal subgroups of a group G, $1 < |C_K(H)| - |C^{\otimes}_K(H)|$ and p the smallest prime divisor of $\lfloor |H|/(|C_K(H)| - |C^{\otimes}_K(H)| - 1)\rfloor$. Then*

$$d^{\otimes}(H,K) \leq d(H,K) - \left(1-\frac{1}{p}\right)\left(\frac{|C_K(H)| - |C^{\otimes}_K(H)| - 1}{|H|}\right).$$

Proof. Using Lemma 4.1, we get

$$d^{\otimes}(H,K) = \frac{1}{|H|}\sum_{i=1}^{k_K(H)} \frac{|C^{\otimes}_K(h_i)|}{|C_K(h_i)|}$$

but $|C^{\otimes}_K(h_i)| \leq |C_K(h_i)|$ for all $h_i \in H$ and $i = 1, \ldots, k_K(H)$ and so we may upper bound the above quantity by

$$\leq \frac{1}{|H|} \underbrace{(1+1+\ldots+1)}_{k_K(H)-\text{times}} = \frac{k_K(H)}{|H|}$$

now we add and subtract a term ad hoc

$$= \frac{k_K(H)}{|H|} + \frac{|C_K(H)| - |C^{\otimes}_K(H)| - 1}{|H|} - \frac{|C_K(H)| - |C^{\otimes}_K(H)| - 1}{|H|}.$$

By assumption, we have $|C_K(H)| - |C^{\otimes}_K(H)| - 1 \geq 1$, and by the choice of p we get

$$\frac{|H|}{|C_K(H)| - |C^{\otimes}_K(H)| - 1} \geq \left\lfloor \frac{|H|}{|C_K(H)| - |C^{\otimes}_K(H)| - 1}\right\rfloor \geq p$$

$$\Rightarrow \frac{|C_K(H)| - |C^{\otimes}_K(H)| - 1}{|H|} \leq \frac{1}{p}.$$

Therefore we may continue our upper bound in the following way:

$$\frac{k_K(H)}{|H|} + \frac{|C_K(H)| - |C^{\otimes}_K(H)| - 1}{|H|} - \frac{|C_K(H)| - |C^{\otimes}_K(H)| - 1}{|H|}$$

$$\leq \frac{k_K(H)}{|H|} + \frac{1}{p} - \frac{|C_K(H)| - |C^{\otimes}_K(H)| - 1}{|H|}$$

and, a fortiori, we get

$$\leq \frac{k_K(H)}{|H|} + \frac{1}{p}\left(\frac{|C_K(H)| - |C^{\otimes}_K(H)| - 1}{|H|}\right) - \frac{|C_K(H)| - |C^{\otimes}_K(H)| - 1}{|H|}$$

$$= d(H,K) - \left(1-\frac{1}{p}\right)\left(\frac{|C_K(H)| - |C^{\otimes}_K(H)| - 1}{|H|}\right).$$

□

Now we will justify the monotonocity of the sequence $d^{\otimes}(H,K)$ in the variable H (and with $K=G$ fixed). At the same time, the following result provides a lower bound in terms of index, once an appropriate sequence of normal subgroups is present.

Theorem 5.2. *Let $H_1, H_2, \ldots, H_m$ be normal subgroups of a group G such that $H_1 \subseteq H_2 \subseteq \ldots \subseteq H_{m-1} \subseteq H_m \subseteq G$ and $i,j \in \{1,\ldots,m\}$ with $i \leq j$. Then $(1/|H_j|)\ d^{\otimes}(H_i,G) \leq |H_j : H_i|\ d^{\otimes}(H_j,G)$ and*

$$\left(\prod_{i=2}^{m} |H_i|\right)^{-1} |H_m : H_1|^{-1}\ d^{\otimes}(H_1,G) \leq \left(\prod_{i=3}^{m} |H_i|\right)^{-1} |H_m : H_2|^{-1}\ d^{\otimes}(H_2,G) \leq \ldots$$

$$\ldots \leq |H_m|^{-1}\ |H_m : H_{m-1}|^{-1}\ d^{\otimes}(H_{m-1},G) \leq d^{\otimes}(H_m,G).$$

Proof. We begin to prove that $(1/|H_j|)\ d^{\otimes}(H_i,G) \leq |H_j : H_i|\ d^{\otimes}(H_j,G)$. Since $H_i \subseteq H_j$ and $C^{\otimes}_{H_i}(g) \subseteq C^{\otimes}_{H_j}(g) \cap H_i \subseteq C^{\otimes}_{H_j}(g)$ for all $g \in G$, we have $C^{\otimes}_{H_i}(g) \subseteq C^{\otimes}_{H_j}(g)$ and so

$$\frac{1}{|C^{\otimes}_{H_j}(g)|} \leq \frac{1}{|C^{\otimes}_{H_i}(g)|} \leq \frac{|H_j|}{|C^{\otimes}_{H_i}(g)|}.$$

Multiplying the right term by $1 = |H_i|/|H_i|$, we rewrite it as

$$\frac{1}{|C^{\otimes}_{H_j}(g)|} \leq \frac{|H_i|}{|C^{\otimes}_{H_i}(g)|}\ \frac{|H_j|}{|H_i|} \quad \Rightarrow \quad \frac{|C^{\otimes}_{H_i}(g)|}{|H_i|} \leq \frac{|H_j|^2}{|H_i|}\ \frac{|C^{\otimes}_{H_j}(g)|}{|H_j|}.$$

With this in mind, it is now clear that

$$d^{\otimes}(H_i,G) = \frac{1}{|G|}\sum_{g\in G}\frac{|C^{\otimes}_{H_i}(g)|}{|H_i|} \leq \frac{1}{|G|}\sum_{g\in G}\frac{|H_j|^2}{|H_i|}\ \left(\frac{|C^{\otimes}_{H_j}(g)|}{|H_j|}\right)$$

$$= \frac{|H_j|^2}{|G|\ |H_i|}\sum_{g\in G}\frac{|C^{\otimes}_{H_j}(g)|}{|H_j|} = \frac{|H_j|^2}{|H_i|}\ \ d^{\otimes}(H_j,G).$$

The first part of the result follows.

Now we want to iterate the inequality $(1/|H_j|)\ d^{\otimes}(H_i,G) \leq |H_j : H_i|\ d^{\otimes}(H_j,G)$ in our situation. The first step gives

$$\frac{|H_1|}{|H_2|^2}\ d^{\otimes}(H_1,G) \leq d^{\otimes}(H_2,G),$$

but we have also

$$\frac{|H_2|}{|H_3|^2}\ d^{\otimes}(H_2,G) \leq d^{\otimes}(H_3,G)$$

and both the inequalities imply

$$\frac{|H_1|}{|H_2|\ |H_3|^2}\ d^{\otimes}(H_1,G) \leq \frac{|H_2|}{|H_3|^2}\ d^{\otimes}(H_2,G) \leq d^{\otimes}(H_3,G),$$

which is equivalent to

$$(|H_2|\ |H_3|)^{-1}\ |H_3 : H_1|^{-1}\ d^{\otimes}(H_1,G) \leq |H_3|^{-1}\ |H_3 : H_2|^{-1}\ d^{\otimes}(H_2,G) \leq d^{\otimes}(H_3,G).$$

After $(m-1)$ steps we find the formula in the statement. The result follows. □

With respect to direct products there is a sort of natural splitting for $d^{\otimes}(H,K)$ and this is shown below. This property was noted in [5, 6, 7, 8, 9] for the relative commutativity degree and in [11, 13] for the relative exterior degree.

Theorem 5.3. *If A, B, C, D are subgroups of a group G such that $(|A|, |B|) = (|C|, |D|) = 1$, then $d^{\otimes}(A \times B, C \times D) = d^{\otimes}(A, C) \cdot d^{\otimes}(B, D)$.*

Proof.
$$|A \times B|\ |C \times D|\ d^{\otimes}(A \times B, C \times D) = |A|\ |B|\ |C|\ |D|\ d^{\otimes}(A \times B, C \times D)$$
$$= \sum_{(a,b)\in A\times B} |C^{\otimes}_{C\times D}((a,b))| = \left(\sum_{a\in A} |C^{\otimes}_{C}(a)|\right) \left(\sum_{b\in B} |C^{\otimes}_{D}(b)|\right)$$
$$= |A|\ |C|\ d^{\otimes}(A, C)\ |B|\ |D|\ d^{\otimes}(B, D).$$
□

Theorem 5.3 is showing that the relative tensor degree satisfies a standard condition of probability measures over finite groups, namely, that the probability of a product equals the product of the probabilities. Another standard condition, encountered in [5, 6, 7, 8, 9], deals with the formation of quotients.

Theorem 5.4. *If H and K are two normal subgroups of G containing a normal subgroup N of G, then $d^{\otimes}(H, K) \leq d^{\otimes}(H/N, K/N)$, and the equality holds whenever $N \subseteq C^{\otimes}_{K}(H)$.*

Proof.
$$|H|\ |K|\ d^{\otimes}(H, K) = \sum_{h\in H} |C^{\otimes}_{K}(h)| = \sum_{hN\in H/N} \sum_{n\in N} |C^{\otimes}_{K}(hn)|$$
$$= \sum_{hN\in H/N} \sum_{n\in N} \frac{|C^{\otimes}_{K}(hn)N|}{|N|}\ |C^{\otimes}_{K}(hn)\cap N| \leq \sum_{hN\in H/N} \sum_{n\in N} |C^{\otimes}_{K/N}(hN)|\ |C^{\otimes}_{K}(hn)\cap N|$$
$$= \sum_{hN\in H/N} |C^{\otimes}_{K/N}(hN)| \sum_{n\in N} |C^{\otimes}_{K}(hn) \cap N| \leq |N|^2 \sum_{hN\in H/N} |C^{\otimes}_{K/N}(hN)|$$
$$= |H|\ |K|\ d^{\otimes}(H/N, K/N).$$
We find always an exact sequence
$$N \otimes K \xrightarrow{\ \varphi\ } H \otimes K \xrightarrow{\ \varepsilon\ } (H/N) \otimes (K/N) \longrightarrow 1$$
where $\iota : n \in N \mapsto \iota(n) \in H$ is the natural embedding of N into H,
$$\varphi : n \otimes k \in N \otimes K \mapsto \iota(n) \otimes k \in H \otimes K$$
and
$$\varepsilon : h \otimes k \in H \otimes K \mapsto hN \otimes kN \in (H/N) \otimes (K/N)$$
is induced by the natural epimorphisms of H onto H/N and of K onto K/N. If $N \subseteq C^{\otimes}_{K}(H)$, then $\mathrm{Im}\,\varphi = 1$ and the previous exact sequence implies $H/N \otimes K/N \simeq H \otimes K$ so that
$$|N|^2\ |\{(hN, kN) \in H/N \times K/N \mid hN \otimes kN = 1\}|$$
$$= |\{(h, k) \in H \times K \mid h \otimes k = 1\}|,$$
hence $d^{\otimes}(H, K) = d^{\otimes}(H/N, K/N)$.

Conversely, assume that $d^{\otimes}(H, K) = d^{\otimes}(H/N, K/N)$. From the equality, we have $|C^{\otimes}_{K}(h) \cap N| = |N|$ for all $h \in H$, and so $N \subseteq C^{\otimes}_{K}(h)$ for all $h \in H$. It implies that $N \subseteq C^{\otimes}_{K}(H)$, as required. □

Example 5.5. Since $Q_{2^n}/Z(Q_{2^n}) \simeq D_{2^{n-1}}$ for all $n \geq 3$, the values of Examples 4.8 and 4.9 may be combined in order to confirm the bound in Theorem 5.4.

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