ON THE WGSC PROPERTY IN SOME CLASSES OF GROUPS

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Abstract. The property of quasi-simple filtration (or qsf) for groups has been introduced in literature more than 10 years ago by S. Brick. This is equivalent, for groups, to the weak geometric simple connectivity (or wgsc). The main interest of these notions is that there is still not known whether all finitely presented groups are wgsc (qsf) or not. The present note deals with the wgsc property for solvable groups and generalized FC-groups. Moreover, a relation between the almost-convexity condition and the Tucker property, which is related to the wgsc property, has been considered for 3-manifold groups.

1. Introduction and terminology

Among simply connected manifolds, universal coverings of compact manifolds are characterized by the existence of a discrete, co-compact and proper discontinuous group action. Furthermore, among contractible open topological n-manifolds, Euclidean spaces can be characterized by a topological property: the *simple connectivity at infinity* (which means, roughly speaking, that loops "at" infinity bound disks "near" the infinity). This is a celebrated result by L.C. Siebenmann [13] for $n \geq 5$ (and by Wall and Freedman in dimension 3 and 4). A similar earlier result was proved by Stallings in [14], but using a stronger definition of the sci.

It has been conjectured for a long time that contractible universal coverings of compact manifolds were homeomorphic to $\mathbb{R}^n$ (or, equivalently, simply connected at infinity). Only in the 80's M. Davis came up with examples contradicting the conjecture in any dimension $n \geq 4$. Then it is meaningful to ask if there exists a topological property which characterizes contractible universal coverings of compact manifolds. A possible candidate comes from the work of A. Casson and V. Poenaru on the previous conjecture in dimension 3 (see [11]). Poenaru's main ingredients are the notions of *geometric simple connectivity* (i.e. handlebody decomposition without 1-handles) and *Dehn-exhaustibility* for open manifolds (see [11]). The latter condition was then slightly modified and adapted to finitely presented groups by S. Brick and M. Mihalik in [1] as follows:

Definition 1.1. A PL-space X is *quasi simply filtered* (or qsf) if for any compact $C \subset X$ there exists a simply connected polyhedron K and a PL-map $f : K \to X$ such that $C \subset f(K)$ and $f|_{f^{-1}(C)} : f^{-1}(C) \to C$ is a PL-homeomorphism.

The finitely presented group G is *qsf* if there exists a polyhedron P whose fundamental group is G and whose universal covering $\widetilde{P}$ is qsf.

Recently, L. Funar and D. E. Otera studied a related notion in [4]:

Definition 1.2. A polyhedron is *weakly geometrically simply connected* (or wgsc) if it admits an exhaustion by compact and simply connected sub-polyhedra.

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A finitely presented group G is *wgsc* if there exists a polyhedron P whose fundamental group is G and whose universal covering $\widetilde{P}$ is wgsc.

Unlike the qsf, the wgsc property depends on the choice of P and on the presentation of the group. However, one can prove that the qsf, the geometric simple connectivity and the wgsc are equivalent for groups (see [4, Corollary 3.1]).

A significant source of examples is given by the following statement, which gives information on several known classes of groups. See [1], [2] and [8] for details.

Theorem 1.3 (S. Brick - M. Mihalik)**.** *Let G be a finitely presented group.*

(1) *G is qsf if and only if a finite index subgroup H of G is qsf.*
(2) *Let A and B be qsf groups and C be a common finitely generated subgroup. The amalgamated free product $D = A *_C B$ is qsf. Moreover, if ϕ denotes an automorphism of A, then the HNN-extension $E = A *_C \phi$ is qsf.*
(3) *Automatic, CAT(0), combable, (semi)-hyperbolic, one-relator groups are qsf.*
(4) *Assume that $1 \to A \to G \to B \to 1$ is a short exact sequence of infinite finitely presented groups. Then G is qsf.*

The main problem is that no example of non-wgsc group is known.

1.1. **Wgsc property and generalized FC-groups.** The next result, whose proof can be found in [4, §4.3], was the beginning of our studies. See also [7].

Proposition 1.4. *Let G be a finitely presented solvable group of derived length $n \geq 1$.*

(1) *Assume that $G^{(n)}$ is finite. G is wgsc if and only if $G/G^{(n)}$ is wgsc.*
(2) *If $G^{(n)}$ contains an element of infinite order, then G is wgsc.*
(3) *If $Z(G)$ is finitely generated, then G is wgsc.*

Finitely presented solvable groups which satisfy some classical chain conditions as *max* and *min* (see [6, Chapter 5] or [12, Chapter 3]) are described with respect to wgsc as follows.

Proposition 1.5. *Let G be a finitely presented solvable group.*

(a) *If G satisfies max (respectively max-sn, max-ab, max-snab), then G is wgsc.*
(b) *If G satisfies min (respectively min-sn, min-ab, min-snab), then G is wgsc.*

Proof. (a). From [12, Theorem 3.31], G is a polycyclic group with derived length $n \geq 1$. For $n = 2$, either G' is finite cyclic or G' is infinite cyclic. In the first case, either G/G' is finite cyclic or G/G' is infinite cyclic. In both cases they are wgsc so that G is wgsc from Proposition 1.4 (1). Iterating this argument, G is wgsc.

If G satisfies either *max-sn*, *max-ab* or *max-snab*, then it is a polycyclic group, by [12, p.176, (v), Theorem 3.31]. The previous argument can be applied.

(b). From [12, Theorem 3.32], G is a Chernikov group, that is, a finite extension of an abelian group with *min*. Since it is finitely generated, then it is a finite group, and hence wgsc.

If G satisfies either *min-sn*, *min-ab* or *min-snab*, then it is again a Chernikov group, respectively by [12, p.176, (v)], [12, Theorem 3.32], [12, p.176, (v)]. □

It is still not clear whether solvable groups with *min-n* or *max-n* are wgsc. However, we are able to prove the wgsc for some classes of generalized FC-groups.

Following [3] and [12, Chapter 4], a group G is called *FC-group*, or group with finite conjugacy classes, if $G/C_G(\langle x\rangle^G)$ is a finite group for each element x of G, where $\langle x\rangle^G$ denotes the normal closure of $\langle x\rangle$ in G and $C_G(\langle x\rangle^G)$ denotes the centralizer of $\langle x\rangle^G$ in

G. The first generalization of the notion of FC-group has been introduced in [10] where the author defines a group G to be a *CC-group*, or group with Chernikov conjugacy classes, if $G/C_G(\langle x\rangle^G)$ is a Chernikov group for each element x of G. Clearly, an FC-group is a CC-group. In [3] the dual notion of CC-group has been introduced: a group G is a *PC-group*, or group with polycyclic-by-finite conjugacy classes, if $G/C_G(\langle x\rangle^G)$ is a polycyclic-by-finite group for each element x of G. Again, an FC-group is a PC-group.

The well known notions of Chernikov group and polycyclic-by-finite group can be extended in the following way. A group A is called *abelian minimax* if it is abelian and satisfies either *max* or *min* (see [6, Chapter 5] or [12, Chapter 10]). A group G is said to be *solvable minimax* if it is solvable and each factor of its derived series is abelian minimax. Furthermore, if the largest normal periodic divisible subgroup of G is trivial, G is said to be *reduced solvable minimax*. This means that G is a finite extension of a solvable group in which each factor of the derived series is torsion-free. Finally, a group G is said to be (*solvable minimax*)-*by-finite* if it is a finite extension of a group which is solvable minimax.

Combining the notions of CC-group and PC-group, L. Kurdachenko defines in [5] a group G to be an *MC-group*, or group with (solvable minimax)-by-finite conjugacy classes, if $G/C_G(\langle x\rangle^G)$ is a (solvable minimax)-by-finite group for each element x of G. Clearly, an FC-group is an MC-group, but also a CC-group is an MC-group, and a PC-group is an MC-group. Finally, a group G is called *M_rC-group*, or group with reduced solvable minimax conjugacy classes, if $G/C_G(\langle x\rangle^G)$ is a reduced solvable minimax group for each element x of G.

Proposition 1.6. *Let G be a finitely presented group.*
(a) *If G is an FC-group, then G is wgsc.*
(b) *If G is a CC-group, then G is wgsc.*
(c) *If G is a PC-group, then G is wgsc.*
(d) *If G is an M_rC-group, then G is wgsc.*

Proof. (a). The structure of a finitely generated FC-group has been described in [12, Corollary 2, p.122]. G is a finite extension of a finitely generated subgroup of its center. We deduce that G is wgsc from Theorem 1.3 (1).

(b). It is easy to see that finitely generated CC-groups are finitely generated FC-groups [12, Theorem 4.36] so that the result follows from the previous step.

(c). A finitely generated PC-group is a finite extension of polycyclic group [3, Theorem 2.2]. Now, a polycyclic group is wgsc from Proposition 1.5 (a), and a finite extension of a wgsc group is obviously wgsc.

(d). First we observe that a reduced solvable minimax group is wgsc. This follows from Proposition 1.4 (1) and (2). Since a finitely presented M_rC-group is a reduced solvable minimax group by [5], the result follows. □

Remark 1.7. A periodic FC-group G can be covered with a family of finite subgroups H_i, where $H_i \subset H_{i+1}$ for each $i \in I$ (see [12, Theorem 4.32 (i)]). Analogously, since G is wgsc, some Cayley 2-complex of G can be covered by means of an ascending family of connected, compact and simply-connected subspaces.

1.2. **Constructible and coherent groups.** Now we study a particular class of finitely presented solvable groups, which has been introduced by G. Baumslag and R. Bieri [6, Chapter 11].

Definition 1.8. Let n be an integer and $0 \leq i \leq n$. A group G is said to be constructible if there is a finite sequence of groups $1 = G_0, G_1, \ldots, G_n = G$ in which each G_i is a finite extension of the fundamental group of a finite graph of groups, whose edge and vertex groups occur among $G_0, G_1, \ldots, G_{i-1}$.

Proposition 1.9. *If G is a constructible group, then G is wgsc.*

Proof. Assume that the constructible group G arises from the sequence of groups $1 = G_0, G_1, \ldots, G_n = G$. For $n = 1$, G_1 is finite and hence wgsc. By induction on n, assume that $n > 1$ and G_i is wgsc for any $i \leq n-1$. The graph Γ_n, which is related to G_n, is a finite graph of groups whose edge and vertex groups occur among the wgsc groups $G_0, G_1, \ldots, G_{n-1}$. Thus G_n is a finite extension of the fundamental group of Γ_n. But the fundamental group of Γ_n arises from the trivial group by forming successive finite extensions, generalized free products and HNN-extensions (in a finite number). Then, by Theorem 1.3 (2), it is wgsc, as claimed. □

Definition 1.10. A group G is called coherent if every finitely generated subgroup is finitely presented.

Proposition 1.11. *Finitely presented solvable coherent groups are wgsc.*

Proof. The characterization [6, 11.4.4] shows how a coherent group can be related to a constructible group. More precisely, a finitely generated solvable group G is coherent if and only if every finitely generated subgroup of G is constructible. Furthermore, this is equivalent to require that G is either polycyclic or an ascending HNN-extension with polycyclic base group [6, 11.4.4]. In the first case, the result follows from Proposition 1.5 (a). In the second case, from Proposition 1.4 (1). □

Corollary 1.12. *A finitely presented nilpotent-by-cyclic group is wgsc.*

Proof. From [6, 11.4.5], G is either polycyclic or an ascending HNN-extension with polycyclic base group. As in the last proposition, we have that G is wgsc. □

Example. Consider the group $G = \mathbb{Z} \times A_5$, where A_5 is the alternating group on 5 elements. G is an FC-group which satisfies Proposition 1.6 (a), and obviously, Proposition 1.6 (b) and (c). This follows by [12, Corollary 2, p.122]. But, G is not a solvable group, since A_5 is simple. Then Proposition 1.4, 1.5 and 1.11 don't hold.

More generally, the same considerations are true for the group $D = A \times B$ instead of G, where A is a finitely generated abelian group and B is the direct product of finitely many non-abelian simple groups.

Remark 1.13. We get to polycyclic-by-finite groups in the proofs of Proposition 1.6 (a), (b) and (c). This is not true for Proposition 1.6 (d). Consider $G = \mathbb{Q}_\pi \rtimes \langle x \rangle$, where $\mathbb{Q}_\pi$ is the group of π-adic rationals and $\langle x \rangle$ has order 2, acting on $\mathbb{Q}_\pi$ by inversion. See [6, Chapter 11, pp.240–241]. G is not polycyclic-by-finite, but is finitely presented and reduced solvable minimax of derived length $d = 2$. G has trivial center, is neither an FC-group, nor a CC-group, nor a PC-group (see [6, Chapter 11, pp.240–241]). Then it satisfies only (d) in Proposition 1.6. It satisfies Propositions 1.9, 1.11 and 1.12, but not 1.5. G is a one-relator group as in Theorem 1.3 (3). More generally, the same is true for $S = \mathbb{Q}_\pi^{d-1} \rtimes \langle x \rangle$, where $d \geq 2$.

2. A relation with the Tucker property

A notion which is related to the wgsc is the *Tucker property*. Originally introduced by W. Tucker for 3-manifolds, the Tucker property was then defined for groups by M. Mihalik and S. Tschantz in [8], where they proved that a Tucker group is qsf.

Definition 2.1. A polyhedron M has the Tucker property if for every subpolyhedron $K \subset M$, each component of $M - K$ has finitely generated fundamental group.

While the wgsc property is related to the existence of a handlebody decomposition without 1-handles, the Tucker property can be interpreted as the existence of a handlebody decomposition with only finitely many 1-handles (see [9]).

Following [9, §4.3], we present here an alternative and shorter proof of the main Theorem of [11], bypassing Poenaru's techniques and the qsf-theory.

We start with a definition. A group G is said *k-almost convex* ($k \geq 2$), if there exists an integer $N = N(k)$ such that any two vertices x, y of the sphere (centered at the identity) of radius r in some Cayley graph of G with $d(x, y) \leq k$, can be joined by a path of length $\leq N$ in the ball of radius r centered at the identity (where the ball and the sphere of radius r centered at the identity e are respectively $B(r) = \{x | \ d(x, e) \leq r\}$ and $S(r) = \{x | \ d(x, e) = r\}$).

Notice that being k-almost convex depends on the presentation of the group; however a group is k-almost convex if and only if it is 2-almost convex (see [11] and the references therein). Thus one can define a group to be *almost-convex* without specifying k.

Proposition 2.2. *Let G be an almost-convex group. Then G has Tucker property.*

Proof. Since G is almost-convex, one can find a presentation of G whose Cayley graph is 3-almost convex. Hence there exists a Cayley 2-complex X of G which is 3-almost convex. Denote by $B(r)$ and $S(r)$ the ball and the sphere of radius r centered at the identity of X. We want to prove that for any r there exists an integer $R = R(r)$ such that any edge-loop in $X - B(r)$, based in $B(r + 1) - B(r)$, is homotopic with respect to the basepoint to a loop in $B(R)$, by an homotopy in $X - B(r)$. This will implies that the inclusion of $B(R) - B(r)$ into $X - B(r)$ induces a surjective homomorphism between their fundamental groups. In particular $\pi_1(X - B(r)$ is finitely generated. Then we can pass to arbitrary finite subcomplex C.

Recall that the isodiametric function of a group is the infimal $I_G(q)$ so that loops of length q bound disks of diameter at most $I_G(q)$ in the Cayley complex. Set $I = I_G(N+2)$ and $R = 2(r + N + I)$, where N is the constant $N(3)$ for X (which is 3-almost convex). Let l be an arbitrary edge-loop in $X - B(r)$ based in $B(r + 1) - B(r)$. Denote by D the distance from l to the identity of X. Let $v_1, ..., v_n$ be those vertices of l realizing the distance D from the origin.

Observe that it is sufficient to show that l can be homotoped, by an homotopy in $X - B(r)$, to a loop l_1 in $B(D - 1)$. In fact, doing this $D - R$ times we can homotope l to a loop l_R in $B(R)$, as wanted. Note that $D > R$ since $l \subset X - B(r)$.

Let $n = 1$. If v_1 has only one adjacent vertex, a, in l, then l is homotopic to the loop $l' \subset B(R)$, where l' is l without the edge $[a, v_1]$. If l has more vertices adjacent to v_1, say $a = a_1, a_2, \ldots, a_t = b$, then we have that a and b lie in the sphere $S(D - 1)$ with $d(a, b) \leq 2$. Since X is 3-almost-convex, there exists a path $\gamma_{a,b}$ of length $\leq N$, contained in the ball of radius $D-1$, joining a and b. Hence the loop consisting of this path followed by the edges $[a, v_1]$ and $[v_1, b]$, has length $\leq N + 2$. By definition of I, any loop of length $\leq N+2$ bounds a disk within I from the loop. This implies that the path (a, v_1, b) can be homotoped to the path $\gamma_{a,b}$ inside the ball of radius $D - 1$, by an homotopy of $X - B(r)$, as wanted.

Suppose $n = 2$ and denote by a (respectively b) the vertices of the loop l sitting in the sphere of radius $D - 1$, adjacent to v_1 (respectively v_2). Then, whenever $a \neq b$, we have that $d(a, b) \leq 3$, and so the same technique applies. While if $a = b$, the loop l is a wedge of a loop $l' \subset B(R)$ with the loop (a, v_1, v_2, b) which sits outside $B(R)$. This loop has length 3 and then it bounds a disk within I. Hence it is null-homotopic within $X - B(r)$. This implies that l is homotopic with respect to the basepoint to $l' \subset B(R)$ by an homotopy in $X - B(r)$.

Whenever $n \geq 3$, our loop has n adjacent vertices, $v_1, \ldots, v_n$ at distance D from e, and we can do the following. Denote by a (respectively b) the vertices of l adjacent to v_1 (respectively v_n) sitting in the sphere of radius $D - 1$, and let p_i be the vertex in the sphere $S(D-1)$ at distance 1 from v_i, for some $i \in [2, n-1]$. Write the edge-loop l as $[e, a_{-m}, \ldots, a_1, a, v_1, \ldots, v_n, b, b_1, \ldots, b_s, e]$. We can apply the inductive hypothesis for the loops $[e, a_{-m}, \ldots, a, v_1, \ldots, v_i, p_i, e]$ and $[e, p_i, v_i, \ldots, v_n, b, b_1 \ldots, b_s, e]$ (in both paths there are less then n vertices at maximal distance). Hence the paths $(a, v_1, \ldots, v_i, p_i)$ and $(p_i, v_i, \ldots, v_n, b)$, as well as $(a, v_1, \ldots, v_n, b)$, can be pushed inside the ball $B(R)$ by an homotopy in $X - B(r)$. This completes the induction and ends the proof. □

Corollary 2.3. *Let M be a closed, irreducible 3-manifold, such that $\pi_1(M)$ is almost-convex and infinite. Then the universal cover $\widetilde{M}$ of M is homeomorphic to $\mathbb{R}^3$.*

Proof. Proposition 2.2 implies that the Cayley 2-complex corresponding to some presentation of $\pi_1(M)$ is Tucker. Being Tucker depends only on the group and not on the presentation [8, Theorem 1], hence $\widetilde{M}$ is Tucker too. From [15, Theorem 1], we have that $\widetilde{M}$ is a missing boundary manifold.[1] Specifically $\widetilde{M} = Int(N^3)$ since it is the universal cover of a closed manifold (and hence without boundary). Moreover N^3 is simply connected too, and then it is a homotopy 3-ball. Finally N^3 is irreducible since $Int(N^3) = \widetilde{M}$ is. Hence N^3 is a 3-ball, since it does not contain "fake 3-balls". This implies that $\widetilde{M}$ is homeomorphic to $\mathbb{R}^3$, as claimed. □

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