On a result of L.-C. Kappe and M. Newell

Francesco Russo[1]

[1]Department of Mathematics ”R.Caccioppoli”,
University of Naples ”Federico II”,
Naples, via Cinthia 80126, Italy.
E-mail: francesco.russo@dma.unina.it

ABSTRACT

There is a long line of research investigating upper central series of a group. The interest comes from the information which these series can give on the structure of a group. Baer (1952) extended the usual notion of center of a group, introducing that of p-centre, where p is a prime. Almost 40 years later, Kappe and Newell (1989) were able to embed the p-centre of a metabelian p-group in the p-th term of the upper central series. This was possible because of the growing knowledge on Engel groups of the 60s years. Here we extend the result of Kappe and Newell (1989) to wider classes of groups.

Keywords: p-hypercentral groups, hypercentral groups, metabelian p-groups.

2000 Mathematics Subject Classification: 20F45, 20F12.

1 Introduction and Statement of Results

Let G be a group, $x_1, \ldots, x_m \in G$, m be a positive integer and $w = w(x_1, \ldots, x_m)$ be a word on G. For an integer i between 1 and m and for an integer $k \neq i$ between 1 and m, define the following set

$$Z(G; w, i) = \{a \in G : w(x_1, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_m) = 1, \forall x_k \in G\}. \tag{1.1}$$

We consider this construction by looking at some simple examples. Clearly, if $w = [x_1, x_2]$, then $Z(G; w, i)$ is the centre $Z(G)$ of G for $i = 1, 2$. Similarly, if $w = [x_1, x_2, \ldots, x_m]$, then $Z(G; w, 1) = Z_{m-1}(G)$ is the (m-1)-th term of the upper central series of G. In the case when $w = [x_1, {}_k x_2]$ is the k-th $Engel$ $word$, results of Baer (1952, 1953) show that for a finite group G, we have that $Z(G; w, 1) = Z_\infty(G)$ is the hypercentre of G and $Z(G; w, 2) = Fitt(G)$ is the Fitting subgroup for k large enough. In the case when $w = x_2^{-n} x_1^{-n} (x_1 x_2)^n$, being n a positive integer, the group $Z(G; w, i)$ allows us to consider the n-$centre$ $Z(G, n) = Z(G; w, 1) \cap Z(G; w, 2)$ introduced in Baer (1952, 1953). All the ”centres” defined above are known to be (characteristic) subgroups of G. Note that $Z(G; w, i)$ is not necessarily a subgroup of G (i.e.: $Z(G; [x, y, y], 2)$ is the set of all left 2-Engel elements of G).

Following Baer (1952), two elements x, y in a group G n-$commute$ if

$$(xy)^n = x^n y^n \ \text{ and } \ (yx)^n = y^n x^n. \tag{1.2}$$

A group is n-$abelian$ if any two elements n-commute. Baer (1952) introduced the n-$centre$ $Z(G, n)$ of a group G as the set of those elements which n-commute with every element in the group. Therefore,

$$Z(G, n) = \{x \in G : (xy)^n y^{-n} x^{-n} = (yx)^n x^{-n} y^{-n} = 1, \forall y \in G\}. \tag{1.3}$$

The n-centre of a group has been intensively studied, as in Brandl (1987, 1991), Brandl et al. (1989), Delizia et al. (2006, 2007), Kappe (1986, 1989), Kappe et al. (1988, 1989, 1990), Kappe (1961), Levi (1942), Moravec (2006), Smith (1994), Tortora (2007). In particular, it is known that $Z(G, n)$ can be embedded, under certain conditions, in a suitable term of the upper central series of G, cf. Kappe and Newell (1989, Theorems 4.3, 4.4, 5.1). For instance, we have that $Z(G, 3) \leq Z_3(G)$ for any group G, and $Z(G, p) \leq Z_p(G)$ when G is a metabelian p-group, where p is a prime.

Following Baer (1953), we say that a group G is n-$hypercentral$ if it has a series

$$1 \lhd Z(G, n) = Z_1(G, n) \lhd Z_2(G, n) \lhd \ldots \lhd Z_j(G, n) \lhd Z_{j+1}(G, n) \lhd \ldots, \tag{1.4}$$

where $Z_{j+1}(G, n)/Z_j(G, n) = Z(G/Z_j(G, n), n)$ for each $j = 1, 2, \ldots$ and

$$G = \bigcup_{j \geq 0} Z_j(G, n). \tag{1.5}$$

If there exists a positive integer c such that $G = Z_c(G, n)$, then G is said to be n-$nilpotent$ of class c.

Our main result is the following.

Main Theorem. *Let G be a metabelian p-group, p be a prime and m be a positive integer. If G is p-hypercentral, then G is hypercentral. Moreover, if G is p-nilpotent of class m, then it is nilpotent of class at most mp.*

Terminology and notations of the present article follow Baer (1952, 1953) and Robinson (1972).

2 Proof of Main Theorem

In the beginning we record some results of Kappe and Newell (1989) that will be used in the proof of our main result. Before formulating the lemma, we define

$$R_n(G) = \{x \in G : [x, {}_n y] = 1, \forall y \in G\}$$

to be the set of right n-Engel elements of a group G.

Lemma 2.1. *Let p be a prime and G be a metabelian group.*

(i) If $a \in R_n(G)$ is of order prime to $n!$, then $a \in Z_{n+1}(G)$.

(ii) If $a \in Z_2(G, n)$, then for all $x \in G$

$$[a, x^{n(1-n)}]Z(G, n) = [a^{n(1-n)}, x]Z(G, n) = [a, x]^{n(1-n)}Z(G, n) = \tag{2.1}$$

$$[a^n, x^{1-n}]Z(G,n) = Z(G,n).$$

(iii) Assume that G is a p-group. If for all $a, x \in G$ with $a^p Z(G,p) \in Z(G/Z(G,p))$ we have

$$\prod_{i=1}^{p-1} [a,_i x,_{p-1-i}, a] Z(G,p) = Z(G,p), \tag{2.2}$$

then

$$[a,_s x,_{r-1}, a] Z(G,p) = Z(G,p) \tag{2.3}$$

for all positive integers r, s such that $r + s \geq p$.

(iv) Assume that G is a p-group. If $a \in R_{p-1}(G)$ and $a^p Z(G,p) \in Z(G/Z(G,p))$, then

$$[a,_{p-1-i} x,_i, a] Z(G,p) = Z(G,p) \tag{2.4}$$

for $i = 1, 2, \dots, p-2$ and all $x \in G$.

(v) Let $u, v \in G$ and m be a positive integer. Then

$$[u, v^m] = \prod_{i=1}^{m} [u,_i v]^{\frac{m!}{i!(m-i)!}}; \tag{2.5}$$

$$(uv^{-1})^m = u^m \left(\prod_{0 \leq i+j \leq m}^{m} [u,_i v,_j u]^{\frac{m!}{(i+j+1)!(m-i-j-1)!}} \right) v^{-m}. \tag{2.6}$$

(vi) Assume that G is a p-group. Then $Z(G,p) \subseteq Z_p(G)$.

Proof. This can be found in Kappe and Newell (1989). □

Lemma 2.2. *Let p be a prime and G be a metabelian p-group. Then*

$$Z_2(G,p) = \{a \in R_{2p-1}(G) : [a^p, x] \in Z(G,p), \forall x \in G\} \tag{2.7}$$

and $Z_2(G,p) \subseteq Z_{2p}(G)$.

Proof. Application of (v) Lemma 2.1 with $x, a \in G$ and $m = p$ yields

$$(ax^{-1})^p Z(G,p) = a^p \left(\prod_{0 \leq i+j \leq p}^{p} [a,_i x,_j a]^{\frac{p!}{(i+j+1)!(p-i-j-1)!}} \right) x^{-p} Z(G,p). \tag{2.8}$$

Assume that a is an element of order p^α in $Z_2(G,p)$, where α is a positive integer. Then there exist integers λ, μ such that $\lambda(p-1) + \mu p^{\alpha-1} = 1$. By (ii) Lemma 2.1, we obtain

$$[a^p, x]Z(G,p) = [a^{\lambda p(p-1)+\mu p^\alpha}, x]Z(G,p) = [a^{\lambda p(p-1)}, x]Z(G,p) = \tag{2.9}$$

$$= [a^\lambda, x]^{p(p-1)} Z(G,p) = Z(G,p)$$

for all $x \in G$. Therefore $a^p Z(G,p) \in Z(G/Z(G,p))$ and $([a,x]Z(G,p))^p = Z(G,p)$ so that (2.8) reduces to

$$\prod_{i=1}^{p-1} [a,_i x,_{p-1-i}, a] Z(G,p) = Z(G,p).$$

Applying Lemma 2.1 (iii), we deduce that each factor in the above product belongs to $Z(G,p)$, in particular $[a,_{p-1}x] \in Z(G,p)$. Now, Lemma 2.1 (vi) gives

$$[a,_{p-1}x] \in Z_p(G), \tag{2.10}$$

then

$$[[[a,_{p-1}x],x], \underbrace{x,\ldots,x}_{(p-1)-times}] = [a,_{2p-1}x] = 1, \tag{2.11}$$

and so $a \in R_{2p-1}(G)$. We deduce that

$$Z_2(G,p) \subseteq \{a \in R_{2p-1}(G) : [a^p, x] \in Z(G,p), \forall x \in G\}. \tag{2.12}$$

Conversely, assume $a \in R_{2p-1}(G)$ and $[a^p, x] \in Z(G,p)$, for all $x \in G$. Since G is a p-group, Lemma 2.1 (i) yields $a \in Z_{2p}(G)$.Therefore,

$$Z(G,p) = [[a^p,x], x_1, \ldots, x_{2p-1}]Z(G,p) = [[a,x], x_1, \ldots, x_{2p-1}]^p Z(G,p), \tag{2.13}$$

for all $x_1 Z(G,p), \ldots, x_{2p-1}Z(G,p) \in G/Z(G,p)$. By an inductive argument and the first part of Lemma 2.1 (v), we obtain

$$[a, x_1, \ldots, x_k]^p Z(G,p) = Z(G,p) \tag{2.14}$$

for all $x_1 Z(G,p), \ldots, x_k Z(G,p) \in G/Z(G,p)$ and $k \geq 1$. Thus (2.8) reduces to

$$(ax^{-1})^p Z(G,p) = a^p \prod_{i=1}^{p-2} [a,_i x,_{p-1-i}, a] x^{-p} Z(G,p). \tag{2.15}$$

Now we can apply Lemma 2.1 (iv) and obtain that each factor in the above product of commutators belongs to $Z(G,p)$. Hence $(ax^{-1})^p Z(G,p) = a^p x^{-p} Z(G,p)$. Thus $a \in Z_2(G,p)$.
The fact that $Z_2(G,p) \subseteq R_{2p-1}(G)$ together with G being a p-group implies that

$$Z_2(G,p) \subseteq Z_{2p}(G) \tag{2.16}$$

by Lemma 2.1 (i).

□

Proposition 2.3. *Let p be a prime and G is a metabelian p-group. Then*

$$Z_m(G,p) = \{a \in R_{m(p-1)}(G) : [a^p, x] \in Z_{m-1}(G,p), \forall x \in G\} \tag{2.17}$$

and $Z_m(G,p) \subseteq Z_{mp}(G)$, where m is a positive integer.

Proof. We may repeat step by step the argument in the proof of Lemma 2.2. By induction, the result follows. □

Now we are in position to finish off the proof of our main result.

Proof. Assume that G is not abelian. Then we consider the upper central series of G. We know that $G = \bigcup_{\lambda<\tau} Z_\lambda(G,p)$, where λ is an ordinal and τ is a limit ordinal. If $G = Z_\tau(G,p)$, then the result is obviously true.
Assume that λ is not a limit ordinal. Proposition 2.3 yields $Z_\lambda(G,p) \leq Z_{\lambda p}(G)$ so that

$$G \subseteq \bigcup_{\lambda<\tau} Z_{\lambda p}(G). \tag{2.18}$$

Then $G = \bigcup_{\lambda<\tau} Z_{\lambda p}(G)$ and it is hypercentral of type λp. If G is p-nilpotent of class m, then

$$G = Z_m(G,p) \subseteq Z_{mp}(G), \tag{2.19}$$

and so $G = Z_{mp}(G)$. The result follows. □

References

Baer, R., 1952, *Endlichkeitskriterien für Kommutatorgruppen*, Math. Ann., 124, 161–177.

Baer, R., 1953, *Factorization of n-soluble and n-nilpotent groups*, Proc. Amer. Math. Soc., 4, 15–26.

Brandl, R., 1987, *Infinite soluble groups with the Bell property: a finiteness condition*, Monatsh. Math., 164, 191–199.

Brandl, R., and Kappe L. -C., 1989, *On n-Bell groups*, Comm. Algebra, 17, 787–807.

Brandl, R., 1991, *Just non hypercentral elements and the Bell centre*, Arch. Math., 57, 319–324.

Delizia, C., Moghaddam, M. R. R., and Rhemtulla, A., 2006, *The structure of Bell groups*, J. Group Theory, 9, 117–125.

Delizia, C., Moravec, P., and Nicotera, C., 2007, *Locally graded Bell groups*, Publ. Math. Debrecen, 71, 1–9.

Kappe, L. -C., 1986, *On n-Levi groups*, Arch. Math., 47, 198–210.

Kappe, L. -C., and Morse, R. F., 1988, *Groups with 3-abelian normal closure*, Arch. Math., 51, 104–110.

Kappe, L. -C., 1989, *On Power Margins*, J. Algebra, 122, 337–344.

Kappe, L. -C., and Newell, M. L, 1989, *On the n-centre of a group*. in: Campbell, C. M. et al. (Eds.), Groups St. Andrews 1989, vol.2, Lecture Notes, London Math. Soc. 160, London: Cambridge University Press, pp.339–352.

Kappe, L. -C., and Newell, M. L., 1990, *On the n-Levi and n-Bell center of a group*. in: Proc. 2nd Int. Conf., Bressanone/Italy 1989, Suppl. Rend. Circ. Mat. Palermo, II Ser. 23, Palermo: Universitá di Palermo, pp.173–181.

Kappe, W. P., 1961, *Die A-Norm einer Gruppe*, Illinois J. Math., 5, 187–197.

Levi, F. W., 1942, *Groups in which the commutator operation satisfies certain algebraic conditions*, J. Indian Math. Soc., 6, 87–97.

Moravec, P., 2006, *On power endomorphisms of n-central groups*, J. Group Theory, 9, 519–536.

Robinson, D. J., 1972, *Finiteness conditions and generalized soluble groups*, Springer Verlag, Berlin.

Smith, H., 1994, *On homomorphic images of locally graded groups*, Rend. Sem. Mat. Univ. Padova, 91, 53–60.

Tortora, A., 2007, *Some Properties of Bell Groups*, Ph.D.Thesis, Mathematics Department, University of Salerno, Salerno.