Generalized Minimal Non-FC-Groups

F. Russo

Mathematics Department, University of Naples Federico II
via Cinthia 80126, Naples, Italy
E-mail: francesco.russo@dma.unina.it

Abstract

Minimal non-FC-groups have been largely investigated and their structure is well-known. Minimal non-CC-groups, a first generalization of minimal non-FC-groups, have been investigated more recently and their complete description is still unknown. The present paper deals with generalizations of the notion of minimal non-FC-group in the context of locally nilpotent and locally solvable groups.

1 Terminology

Let $\mathfrak{X}$ be a class of groups. A group belonging to $\mathfrak{X}$ is called $\mathfrak{X}$-*group*. Following [9, 10, 12], $\mathfrak{X}$ is $\mathbf{S}_n$-*closed*, if every normal subgroup of an $\mathfrak{X}$-group is an $\mathfrak{X}$-group. $\mathfrak{X}$ is $\mathbf{N}$-*closed*, if the subgroup generated by any system of normal $\mathfrak{X}$-subgroups of a group is an $\mathfrak{X}$-group. $\mathfrak{X}$ is $\mathbf{P}$-*closed*, if every extension of an $\mathfrak{X}$-group is an $\mathfrak{X}$-group. From now, $\mathfrak{X}$ will be $\mathbf{S}_n$-, $\mathbf{N}$- and $\mathbf{P}$-closed.

An element x of a group G is said to be XC-*central*, if $G/C_G(\langle x\rangle^G)$ is an $\mathfrak{X}$-group (see [10]). An XC-*group* is a group in which all the elements are XC-central. In an XC-group we may introduce the *upper* XC-*central series*, generalizing the usual notion of upper central series of a group. This gives the corresponding notions of XC-*center*, XC-*hypercenter* and XC-*length* of a group. See [2], [8, Section 1], [10, Sections 1, 2], [12, Chapters 4, 5] and [13].

If $\mathfrak{X}$ is the class $\mathfrak{F}$ of all finite groups, we have the notion of FC-hypercentral group. See [12, Chapters 4, 5] and [2, 8]. If $\mathfrak{X}$ is the class $\mathfrak{PF}$ of all polycyclic-by-finite groups, we have the notion of PC-hypercentral group. See [2, 4, 8, 10]. If $\mathfrak{X}$ is the class $\mathfrak{C}$ of all Chernikov groups, we have the notion of CC-hypercentral group. See [2, 8, 10]. If $\mathfrak{X}$ is the class $\mathfrak{S}_2\mathfrak{F}$ of all (solvable minimax)-by-finite groups, we have the notion of MC-hypercentral group. See [6, 7, 10, 13].

We list the following classes of groups, which can be found in [9, Chapter 5].

- $\mathfrak{S}_0$ is the class of all *solvable groups with finite abelian ranks*. $\mathfrak{S}_0\mathfrak{F}$ is the class of all (finite abelian rank solvable)-by-finite groups. An element x of a group G is said to be S_0C-*central*, if $G/C_G(\langle x\rangle^G)$ is an $\mathfrak{S}_0\mathfrak{F}$-group.

- $\mathfrak{S}_1$ is the class of all *solvable groups with finite abelian total rank*. $\mathfrak{S}_1\mathfrak{F}$ is the class of all (finite abelian total rank solvable)-by-finite groups. An element x of a group G is said to be S_1C-*central*, if $G/C_G(\langle x\rangle^G)$ is an $\mathfrak{S}_1\mathfrak{F}$-group.

- $\mathfrak{S}_2$ is the class of all *solvable minimax groups*. $\mathfrak{S}_2\mathfrak{F}$ is the class of all (minimax solvable)-by-finite groups. An element x of a group G is said to be S_2C-*central*, if $G/C_G(\langle x\rangle^G)$ is an $\mathfrak{S}_2\mathfrak{F}$-group. S_2C-*groups*, or MC-*groups*, are described in [7, 13].

- $\mathfrak{S}_3$ is the class of all *solvable groups with finite torsion-free rank*. $\mathfrak{S}_3\mathfrak{F}$ is the class of all (finite abelian torsion-free rank solvable)-by-finite groups. An element x of a group G is said to be S_3C-*central*, if $G/C_G(\langle x\rangle^G)$ is an $\mathfrak{S}_3\mathfrak{F}$-group.

- $\mathfrak{S}_4$ is the class of all *solvable groups with finite Prüfer rank*. $\mathfrak{S}_4\mathfrak{F}$ is the class of all (finite Prüfer rank solvable)-by-finite groups. An element x of a group G is said to be S_4C-*central*, if $G/C_G(\langle x\rangle^G)$ is an $\mathfrak{S}_4\mathfrak{F}$-group.

From [9, Figure 5.1, p.86] and the above definitions, it is clear that the following diagram is true.

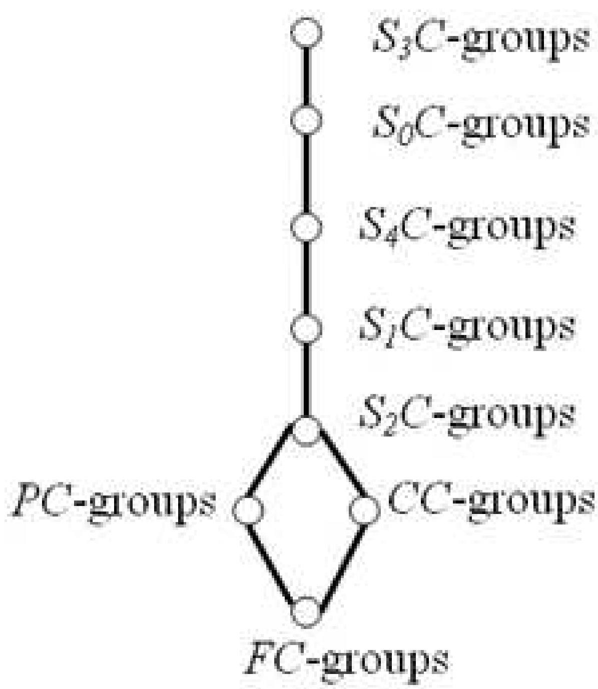

Fig.1.1. A schematic illustration of some classes of generalized FC-groups.

It is instructive to compare Fig.1.1 with [9, Figure 5.1, p.86]. This justifies our natural approach to generalized FC-groups.

Recall that a group G is called *minimal non-FC-group* if it is not an FC-group but all whose proper subgroups are FC-groups. The above considerations allow us to extend such a notion for $i \in \{0,1,2,3,4\}$. More precisely, a group G is called *minimal non-S_iC-group* if it is not an S_iC-group but all whose proper subgroups are S_iC-groups. Looking at the results in [9, Chapter 5], we note that $\mathfrak{S}_i\mathfrak{F}$ coincides with $\mathfrak{C}$ for a periodic group, where $i \in \{0,1,2,3,4\}$. Then minimal non-S_iC-groups are exactly the groups in [3, 11]. See also [14]. Of course, we may illustrate a diagram as Fig.1.1 for the inclusions between minimal non-FC-groups, minimal non-CC-groups, minimal non-PC-groups and minimal non-S_iC-groups, where $i \in \{0,1,2,3,4\}$.

After the preliminary results in Section 2, we describe minimal non-S_iC-groups in Section 3.

2 Preliminaries

The following condition generalizes some circumstances which can be found in [5, Lemma 4, Theorem 1, Theorem 2] for S_0C-groups.

Condition 2.1. *Let $i \in \{0,1,2,3,4\}$ and x be an element of a group G. If G is an S_iC-group, then $\langle x\rangle^G$ is an $\mathfrak{S}_i\mathfrak{F}$-group.*

We know from [5, Theorem 1] that Condition 2.1 is always verified for $i = 0$. This is true also for $i = 2$, since S_2C-groups are a subclass of S_0C-groups. Unfortunately, [5, Example p.150] shows that this is false for $i = 1, 3, 4$. Note that Condition 2.1 defines the so called *Dietzmann classes*. See [10] for details.

Let G be a group. A subgroup H of G is said to be $\mathfrak{F}$-perfect if it has no proper subgroups of finite index (in H). See [12, Vol.2, Sections 9.2, 9.3] or [7]. The subgroup $R(G)$ of G generated by all $\mathfrak{F}$-perfect subgroups of G is clearly $\mathfrak{F}$-perfect. $R(G)$ is called the *$\mathfrak{F}$-perfect part of G*. Obviously, $D(G) \leq R(G)$, where $D(G)$ is the subgroup of G generated by all Chernikov divisible normal subgroups of G.

Remark 2.2. Let G be an S_2C-group. If $D(G)$ is trivial, then $G/Z(G)$ is a residually finite group (see [7, p.226, l.3-9]). Note that this is due to the fact that an $\mathfrak{S}_2\mathfrak{F}$-group G with trivial $D(G)$ has a torsion-free subgroup of finite index. This happens also for $\mathfrak{S}_0\mathfrak{F}$-groups, $\mathfrak{S}_1\mathfrak{F}$-groups, $\mathfrak{S}_3\mathfrak{F}$-groups, $\mathfrak{S}_4\mathfrak{F}$-groups, as shown by [9, 5.2.2, 5.2.3, 5.2.4, 5.2.6, 5.2.9, 5.2.10, 5.2.11, 5.3.1, 5.3.2, 5.3.6, 5.3.7, 5.3.8, 5.3.9, 5.3.10, 5.3.11]. Therefore an S_0C-group G with trivial $D(G)$ has $G/Z(G)$ which is a residually finite group. The same happens for an $\mathfrak{S}_1C$-group, an $\mathfrak{S}_3\mathfrak{F}$-group and an $\mathfrak{S}_4\mathfrak{F}$-group. ■

The following result is mainly due to the previous remark. See also [7, Lemma 2].

Lemma 2.3. *Assume that Condition 2.1 is true and that* $i \in \{0, 1, 2, 3, 4\}$*. If* G *is an* S_iC*-group and* $D(G)$ *is trivial, then* $R(G) \leq Z(G)$*. If* $D(G)$ *is nontrivial, then* $D(G) \leq Z(R(G))$ *and* $R(G)/D(G)$ *is a torsion-free divisible abelian group.*

Proof. Let x be an element of G. Then $R(G)$ is contained in every subgroup H of G of finite index such that $H \geq C_G(\langle x\rangle^G)$. Since $G/C_G(\langle x\rangle^G)$ is a residually finite group, we have that $R \leq C_G(\langle x\rangle^G)$. It follows that

$$R(G) \leq \bigcap_{x\in G} C_G(\langle x\rangle^G) = Z(G),$$

as required. Now assume that $D(G)$ is nontrivial and consider $R(G)/D(G)$. We may repeat the same argument so that $R(G)/D(G) \leq Z(G/D(G))$. Note that the largest periodic subgroup of $R(G)/D(G)$ is divisible abelian so that $R(G)/D(G)$ is torsion-free. If $y \in D(G)$, then $\langle y\rangle^G$ is an $\mathfrak{S}_i\mathfrak{F}$-group by Condition 2.1. Since $D(G)$ is abelian, $\langle y\rangle^G$ is bounded and so finite. Thus, $G/C_G(\langle y\rangle^G)$ is finite, too. We deduce that $R(G) \leq C_G(\langle y\rangle^G)$. As before,

$$D(G) \leq R(G) \leq \bigcap_{y\in D(G)} C_G(\langle y\rangle^G) = Z(D(G)) = D(G) \cap Z(R(G)),$$

from which $D(G) \leq Z(R(G))$, as required. Then the result follows. ∎

Now, we are in position to extend some results on chain conditions in generalized FC-groups as for instance [2, Proposition 6], [4, Theorem 3.2], [8, Corollary 2.3] and [12, Theorems 4.37, 4.38]. We consider mainly two cases: locally nilpotent case and locally solvable case.

Proposition 2.4. *Let* $i \in \{0, 1, 2, 3, 4\}$ *and* G *be a group with trivial* $R(G)$*. If* G *is a locally nilpotent* S_iC*-group, then* G *is hypercentral. If* G *is a locally solvable* S_iC*-group, then* G *is hyperabelian.*

Proof. Assume that G is a locally nilpotent S_iC-group. If $i = 2$, then the result is true by [6, Theorem 3].

Let $i \neq 2$ and consider $G/R(G)$. We note that $G/R(G)$ has no nontrivial divisible normal subgroups and so it is a locally nilpotent PC-group. Applying [4, Theorem 3.2], $G/R(G)$ is hypercentral of type at most ω. Since $R(G)$ is trivial, the result follows.

Assume that G is a locally solvable S_iC-group. If $i = 2$, then the result is true by [6, Theorem 4].

Let $i \neq 2$ and consider $G/R(G)$. We note that $G/R(G)$ has no nontrivial divisible normal subgroups and so it is a locally solvable PC-group. Applying again [4, Theorem 3.2], $G/R(G)$ is hyperabelian of type at most ω. Since $R(G)$ is trivial, the result follows. ∎

Remark 2.5. If Condition 2.1 is true, then for each $i \in \{0,1,2,3,4\}$ an S_iC-group is a locally-(normal and $\mathfrak{S}_i\mathfrak{F}$-group). The converse is true for PC-groups [4, Theorem 2.2] and periodic CC-groups [12, Theorem 4.6]. A nonperiodic CC-group shows that the converse of Condition 2.1 is false. See [10, Example B.8 d]. ■

Lemma 2.6. *Assume that $i \in \{0,1,2,3,4\}$. If $G = HK$ is the product of a normal $\mathfrak{S}_i\mathfrak{F}$-group H by an S_iC-group K, then G is an S_iC-group.*

Proof. Let x be an element of G. Since a quotient of an S_iC-group is an S_iC-group, G/H is an S_iC-group and so $(G/H)/C_{G/H}(\langle xH\rangle^{G/H})$ is an $\mathfrak{S}_i\mathfrak{F}$-group. Therefore $K/C_K(\langle \overline{x}\rangle^K)$ is an $\mathfrak{S}_i\mathfrak{F}$-group, where $\overline{x} = xH$. Since $C_H(\langle x\rangle^H) \cap C_K(\langle \overline{x}\rangle^K) \leq C_{HK}(\langle x\rangle^{HK})$, $G/C_G(\langle x\rangle^G) = (HK)/C_{HK}(\langle x\rangle^{HK})$ is isomorphic to a subgroup of $(HK)/(C_H(\langle x\rangle^H) \cap C_K(\langle \overline{x}\rangle^K))$. But, this is isomorphic to a subgroup of

$$(HK)/C_H(\langle x\rangle^H) \times (HK)/C_K(\langle \overline{x}\rangle^K) \leq \langle H/C_H(\langle x\rangle^H), K/C_K(\langle \overline{x}\rangle^K)\rangle = A.$$

Now, A is an $\mathfrak{S}_i\mathfrak{F}$-group, because $\mathfrak{S}_i\mathfrak{F}$ is $\mathbf{S}_n$-, $\mathbf{N}$- and $\mathbf{P}$-closed. Then x is an S_iC-element of G. The choice of x was arbitrary, then we may conclude that G is an S_iC-group. ■

3 Main results

The present Section deals with the main result of the paper. Note that *minimal non-hypercentral* groups, and indirectly *minimal non-hyperabelian groups*, are known by [1].

Proposition 3.1. *Let $i \in \{0,1,2,3,4\}$ and G be a locally nilpotent group in which every proper subgroup H of G has trivial $R(H)$. If one of the following conditions is true:*

(i) *G is a minimal non-S_iC-group;*

(ii) *G is a minimal non-CC-group;*

(iii) *G is a minimal non-PC-group;*

(iv) *G is a minimal non-FC-group;*

then either G is a hypercentral group or it is a minimal non-hypercentral group.

Proof. Fix $i \in \{0,1,2,3,4\}$ and G be a minimal-non-S_iC-group. From Proposition 2.4, every proper subgroup H of G is hypercentral. The result follows. ■

Proposition 3.2. *Let $i \in \{0,1,2,3,4\}$ and G be a locally solvable group in which every proper subgroup H of G has trivial $R(H)$. If one of the following conditions is true:*

(i) *G is a minimal non-S_iC-group;*

(ii) *G is a minimal non-CC-group;*

(iii) *G is a minimal non-PC-group;*

(iv) *G is a minimal non-FC-group;*

then either G is a hyperabelian group or it is a minimal non-hyperabelian group.

Proof. Fix $i \in \{0, 1, 2, 3, 4\}$ and G be a minimal non-S_iC-group. From Proposition 2.4, every proper subgroup H of G is hyperabelian. The result follows. ■

Proposition 3.3. *Fix $i \in \{0, 1, 2, 3, 4\}$. If G is a minimal non-S_iC-group and Condition 2.1 is true for every proper subgroup of G, then $S_i(G)$ is trivial. In particular, $Z(G)$ is trivial.*

Proof. Lemma 2.6 implies that G has no nontrivial normal subgroups which are $\mathfrak{S}_i\mathfrak{F}$-groups. Assume that $S_i(G)$ is a nontrivial subgroup of G. By assumptions, $S_i(G)$ has to be a proper subgroup of G so it satisfies Condition 2.1. Therefore, $\langle x\rangle^G$ is an $\mathfrak{S}_i\mathfrak{F}$-group, where x is an element of $S_i(G)$. From this, $\langle x\rangle^G$ has to be trivial and so $S_i(G)$ must be trivial. From this contradiction and from the fact that $S_i(G) \geq Z(G)$, the result follows. ■

Remark 3.4. Hypercentral groups, FC-hypercentral groups, PC- hypercentral groups, CC-hypercentral groups and MC-hypercentral groups can be very different as shown by the infinite dihedral group or by examples in [2, 8, 13]. This implies that Propositions 3.2 and 3.3 give strong restrictions for minimal non-S_iC-groups.

Acknowledgement. The author is indebted to Prof. N. Trabelsi for his deep suggestions on the topic.

References

[1] A. O. Asar. Locally Nilpotent p-Groups whose Proper Subgroups are Hypercentral or Nilpotent-by-Chernikov. *J. London Math. Soc.* (2) **61** (2000), 412–422.

[2] J. C. Beidleman, A. Galoppo and M. Manfredino. On PC-hypercentral and CC-hypercentral groups. *Comm. Alg.* **26** (1998), 3045–3055.

[3] B. Bruno and R. E. Phillips. On minimal conditions related to Miller-Moreno type groups. *Rend. Sem. Univ. Padova* **69** (1983), 153–168.

[4] S. Franciosi, F. de Giovanni and M. J. Tomkinson. Groups with polycyclic-by-finite conjugacy classes. *Boll. Un. Mat. Ital.* **7** (1990), 35–55.

[5] L. Kurdachenko. On Normal Closures of Elements in Generalized FC-Groups. In :*Infinite Groups 1994. Proccedings of the International Conference held in Ravello, Italy, May 23-27, 1994*, Eds.: F. de Giovanni and M. Newell, (de Gruyter, Berlin, 1996), pp.141–151.

[6] L. Kurdachenko. *On groups with minimax conjugacy classes.* In : *Infinite groups and adjoining algebraic structures*, (Naukova Dumka, Kiev, 1999), pp.160–177.

[7] L. Kurdachenko and J. Otál. Frattini properties of groups with minimax conjugacy classes. In :*Topics in Infinite Groups*, Ed.: Dip. Mat. Seconda Univ. Napoli, Quad. di Mat. vol. 8, (Aracne, Rome, 2000), pp.223–235.

[8] T. Landolfi. On generalized central series of groups. *Ricerche Mat.* **XLIV** (1995), 337–347, .

[9] J. Lennox and D. J. Robinson. *The Theory of Infinite Soluble Groups*, Oxford University Press, 2004, Oxford.

[10] R. Maier. *Analogues of Dietzmann's Lemma.* In: *Advances in Group Theory 2002*, Eds.: F. de Giovanni and M. Newell, (Aracne, Rome, 2003), pp.43–69.

[11] J. Otál and J. M. Peña. Minimal Non-CC-Groups. *Comm. Algebra* (6) **16** (1988), 1231–1242.

[12] D. J. Robinson. *Finiteness conditions and generalized soluble groups*, vol. I and vol.II, Springer Verlag, 1972, Berlin.

[13] F. Russo. MC-Hypercentral Groups. *Int. J. Contemp. Math. Sciences* (29) **2** (2007), 1441–1449.

[14] F. Russo and N. Trabelsi. On Minimal Non-PC-Groups. To appear.