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A note on just-non-Ω groups

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ABSTRACT. Extending some previous notions in literature, we consider the class of just-non-Ω groups, where Ω is a prevariety of topological groups. Some structure theorems are shown in the compact case. We further analyze some concrete examples.

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1. THE NOTION OF $JN\Omega$ GROUP

All groups which are considered in this paper are topological groups satisfying Hausdorff separation.
A class of topological groups Ω is called a *prevariety* if it is closed under the following conditions for all groups G, H and N:

(0) if $G \in \Omega$ and $G \simeq H$, then $H \in \Omega$

(A) if $G \in \Omega$ and $H = \overline{H} \leq G$, then $H \in \Omega$

(B) if $N = \overline{N}$ is a normal subgroup of G and $N, G/N \in \Omega$, then $G \in \Omega$

(C) if $G, H \in \Omega$, then $G \times H \in \Omega$.

Examples of prevarieties are the class of finite discrete groups and the class of Lie groups.
A prevariety is called a *variety* if it satisfies the following condition:

(C*) if $\{G_j : j \in J\}$ is any family of groups from Ω, then $\prod_{j \in J} G_j \in \Omega$.

The prevariety of finite groups, or that of Lie groups, fail to be varieties, since they don't satisfy (C*). Note that a variety in the traditional sense is referred to [5, 6] and in general cannot satisfy (B), as shown by the abelian groups. However the varieties in our sense exist, since the trivial groups satisfy all the above requirements. On another hand, (0, A, B, C, C*) are independent conditions and so we may always consider the smallest class of topological groups satisfying (0, A, B, C, C*). In this way, we may get a variety from a given prevariety. Then we may construct concrete examples, beginning from finite groups or Lie groups. Note that every prevariety Ω defines a full subcategory of the category of Hausdorff topological groups and continuous group morphisms whose objects are the groups of Ω and whose morphisms are all continuous group morphisms between the objects.

Remark 1.1. *A prevariety of Haudorff topological groups is closed with respect to the formation of finite limits. A variety is closed with respect to the formation of all limits, in particular projective limits.*

Proof. By (A), a prevariety is closed under the formation of equalizers (see [4, A3.43(ii)]) and by (C), it is closed under products of finite families. Hence, by [4, Theorem A3.47], both assertions follow. □

Now we may formulate the following concept which we see of great interest in different contexts in literature. See [1, 2, 7, 10, 11].

Definition 1.2. *Let Ω be a prevariety of Haudorff topological groups. A topological group is called a just-non-Ω group, or briefly a JNΩ group, if it is not in Ω, but all of its proper Hausdorff quotients are in Ω.*

Note that all Haudorff topological simple groups, that is, Haudorff topological groups without proper nonsingleton closed normal subgroups, are $JN\Omega$ groups by default for any choice of the prevariety Ω.

In Definition 1.2, if Ω is a variety, we have a *JNV group*. Compact *JNL* groups were studied in [10], when Ω is the prevariety of Lie groups. We note that [2, Theorems 1.2, 1.4, 1.7 and Corollaries 1.3, 1.5] show similar circumstances for locally compact groups which admit only proper discrete quotient groups. This situation is considered in Definition 1.2 when Ω is the prevariety of compact groups. Similarly, [1, Theorem 9.1] describes just-infinite pro-2 groups, which satisfy Definition 1.2 when Ω is the prevariety of finite groups and we are in the category of pro-2 groups with corresponding morphisms. These examples show the interest in literature for suitable choices of the prevariety Ω.

2. The structure of $JN\Omega$ groups

The present section deals with the groups in Definition 1.2 in the compact case. For a compact group G and a prevariety Ω, $\mathcal{V}(\mathcal{G})$ denotes the set of all normal closed subgroups of G such that $N \in \mathcal{V}(\mathcal{G})$ if and only if $G/N \in \Omega$. See [4, pp.148–149] in the case when Ω is the prevariety of Lie groups, i.e. $\mathcal{V}(\mathcal{G}) = \mathcal{N}(\mathcal{G})$. Since $\{1\}$ is obviously in Ω, $G \in \mathcal{V}(\mathcal{G})$. Furthermore $\{1\} \in \mathcal{V}(\mathcal{G})$ if and only if G is in

Ω. It follows that in a compact $JN\Omega$ group G, the set $\mathcal{V}(\mathcal{G})$ contains all closed normal subgroups of G except $\{1\}$. Then we have the following statements.

Lemma 2.1. *If G is a compact $JN\Omega$ group, then all members $N \in \mathcal{V}(\mathcal{G})$ are not in Ω.*
Proof. This follows from (B) and the definition of $\mathcal{V}(\mathcal{G})$. □

Lemma 2.2. *If G is compact, then $\mathcal{V}(\mathcal{G})$ is closed with respect to intersections and thus is a filter basis.*
Proof. Let $N_1, N_2 \in \mathcal{V}(\mathcal{G})$. Then $G/N_1, G/N_2 \in \Omega$ by definition of $\mathcal{V}(\mathcal{G})$. By (C), we have $G/N_1 \times G/N_2 \in \Omega$. The morphism of topological groups

$$f : g \in G \mapsto f(g) = (gN_1, gN_2) \in G/N_1 \times G/N_2$$

has $\ker f = N_1 \cap N_2$ so that $\mathrm{im} f \simeq G/(N_1 \cap N_2)$. Since G is compact, $\mathrm{im} f$ is compact, hence a closed subgroup of $G/N_1 \times G/N_2$ and thus is in Ω by (A). Now we conclude by (0) that $G/(N_1 \cap N_2) \in \Omega$. Hence $N_1 \cap N_2 \in \mathcal{V}(\mathcal{G})$. The result follows. □

We might say that a topological group G is *residually* Ω, if $\bigcap \mathcal{V}(\mathcal{G}) = \{\infty\}$, i.e., if the quotient morphism $G \to G/N$ for $N \in \mathcal{V}(\mathcal{G})$ separates the points. This notion is well-known in the case of Lie groups, as described in [5, 6]. Indeed, a compact non-Lie $JN\Omega$ group is always residually Ω, since $\mathcal{V}(\mathcal{G}) \subseteq \mathcal{N}(\mathcal{G})$ and G is residually $\mathcal{N}(\mathcal{G})$. These observations allow us to prove the following facts.

Corollary 2.3. *If G is compact and residually Ω, then G is a pro-Ω-group. In particular, a compact non-Lie group is a pro-Ω-group.*
Proof. By Lemma 2.2, the projective limit $\lim_{N \in \mathcal{V}(\mathcal{G})} G/N$ is well-defined, and there is a natural morphism $\phi : G \to \lim_{N \in \mathcal{V}(\mathcal{G})} G/N$ which is injective, because of $\bigcap \mathcal{V}(\mathcal{G}) = \{\infty\}$. Since G is compact, the hypothesis $\bigcap \mathcal{V}(\mathcal{G}) = \{\infty\}$ implies $\lim \mathcal{V}(\mathcal{G}) = \{\infty\}$. If $(g_N N)_{N \in \mathcal{V}(\mathcal{G})} \in \lim_{N \in \mathcal{V}(\mathcal{G})} G/N$ and $N \subseteq M$ in $\mathcal{V}(\mathcal{G})$, then $(g_N^{-1} g_M) \in M$; this means that $(g_N)_{N \in \mathcal{V}(\mathcal{G})}$ is a Cauchy net in G and therefore converges as G is complete as a compact group. If $g = \lim_{N \in \mathcal{V}(\mathcal{G})} g_N$, then $g \in g_N N$ for all N and thus $\phi(g) = (g_N)_{N \in \mathcal{V}(\mathcal{G})}$. In particular ϕ is surjective. Since G is a compact group, ϕ is an isomorphism of topological groups. Hence G is a pro-Ω group. □

Corollary 2.3 has the following consequence.

Corollary 2.4. *A non-Lie compact group cannot be a JNV group.*
Proof. Lemma 1.1 and Corollary 2.3 would give the contradiction $G \in \Omega$. □

Corollary 2.5. *Let G be a compact group. If $N \in \Omega \cap \mathcal{V}(\mathcal{G})$, then $G \in \Omega$. In particular, in a $JN\Omega$ group G, we have $\Omega \cap \mathcal{V}(\mathcal{G}) = \emptyset$.*
Proof. If $N \in \Omega \cap \mathcal{V}(\mathcal{G})$, then $N, G/N \in \mathcal{V}(\mathcal{G})$ and (B) gives $G \in \Omega$. □

Recall that the identity component of a compact group G is denoted G_0 (see [4]).

Remark 2.6. *In a $JN\Omega$ compact group G, if $G_0 \neq \{1\}$, then $G_0 \notin \Omega$.*
Proof. This follows from Lemma 2.1. □

Remark 2.7. *The intersection of two nontrivial normal closed subgroups in a compact $JN\Omega$ group G cannot be trivial.*
Proof. Let N and M be two nontrivial normal closed subgroups in G such that $N \cap M = \{1\}$. Then G is isomorphic, as compact group, to a closed subgroup of $G/N \times G/M$, which is in Ω by (C). This is a contradiction. □

Remark 2.7 shows why we find groups with only one minimal normal subgroup among $JN\Omega$ groups. In many situations in [1, 2, 7, 10, 11] such a subgroup coincides with the *nilradical* $N(G)$ of a compact group G, where $N(G)$ is the largest connected normal nilpotent subgroup of G.

Lemma 2.8. *If G is a compact $JN\Omega$ group, then the nilradical $N(G)$ is either a singleton or an abelian group.*
Proof. We know from [4, Theorem 9.24] that a connected and solvable compact group is abelian. Considering $N(G)$, the result follows. □

The next result can be found in a different way in [4, Chapter 9]. Recall from [4, Definition 6.17] that a connected compact Lie group G is called *simple* if every proper normal subgroup is discrete (and so central). G is called *semisimple* if $\{1\}$ is the only connected central proper subgroup.

Lemma 2.9. *Let G be a compact group and $Z = Z(G_0)_0$. Then G contains a closed subgroup H with a semisimple identity component $H_0 = (G_0)'$ such that $G = ZH$ with $Z \cap H$ totally disconnected and normal in G.*
Proof. By Lee's Supplement Theorem [4, Theorem 9.41], there is a totally disconnected compact subgroup D such that $G = G_0 D$ with $G_0 \cap D \subseteq Z(G_0)$. The commutator subgroup $(G_0)'$ is a closed connected characteristic subgroup of G_0 by Goto Commutator Subgroup Theorem [4, Theorem 9.2]. Hence $H = (G_0)'D$ is a closed subgroup for which $H_0 - (G_0)'$ and D are totally disconnected. Since $G_0 = Z(G_0)'$ by [4, Theorem 9.24], we have $G = G_0 D = ZH$. The normalizer of $Z \cap H$ in G contains Z, since Z is abelian, and it contains H, since Z is characteristic in G_0 and hence normal in G. Hence $Z \cap H$ is normal in G. Finally, we

claim that $Z \cap H$ is totally disconnected. Indeed, $(Z \cap H)_0$ is a connected abelian normal subgroup. Therefore it is contained in H_0, but H_0 is semisimple and thus $(Z \cap H)_0 = \{1\}$. □

The structure of a compact $JN\Omega$ group with nontrivial nilradical is the following.

Theorem 2.10. *Let G be a non-abelian compact $JN\Omega$ group such that $G_0 \neq \{1\}$ is not semisimple. If $N(G) \neq \{1\}$, then $G \simeq N(G) \rtimes V$, where $V \neq \{1\}$ is a closed subgroup in Ω operating effectively on the abelian group $N(G)$. Furthermore, $V_0 = (G_0)'$ is semisimple.*

Proof. The assumption that $G_0 \neq \{1\}$ is not semisimple is done to avoid the case of simple groups, which satisfy obviously the result.

Note that we are in the situation of Lemma 2.9, where $N(G)$ is an abelian group by Lemma 2.8, playing the role of Z. Then $N(G) \cap V$ is totally disconnected and normal in G, where V is a closed subgroup of G with a semisimple identity component $V_0 = (G_0)'$. Furthermore, $G = N(G)V$. Since G is a $JN\Omega$ group, $G/N(G) \simeq V$ is a compact group in Ω. Since G is non-abelian, $N(G) \neq G$ and so $V \neq \{1\}$. On the other hand, $N(G)$ cannot be in Ω, otherwise (B) would give a contradiction. This implies that $N(G) \cap V = \{1\}$ and so $G \simeq N(G) \rtimes V$.

The set of elements of V leaving all of $N(G)$ elementwise fixed is $D = V \cap Z(N(G), G)$, where $Z(N(G), G)$ denotes the centralizer of $N(G)$ in G. The normalizer of D in G contains V and $N(G)$. Thus D is normal in G. We have that V is in Ω, whereas $Z(N(G), G)$ is not in Ω, otherwise (B) would give a contradiction. Hence $D = \{1\}$, that is, V acts effectively on $N(G)$. The result follows. □

Corollary 2.11. *Assume G is a non-abelian compact $JN\Omega$ group such that $G_0 \neq \{1\}$ is not semisimple. If Ω is the prevariety of Lie groups and $G/N(G)$ is abelian, then G splits over the group $\mathbb{Z}_p$ of the p-adic integers for some prime p.*

Proof. From Theorem 2.10, $G = N(G) \rtimes V$, where V is a compact abelian Lie group and $N(G)$ is an abelian compact non-Lie group. From the structure of compact abelian Lie groups [4, Proposition 2.42], we know that $V \simeq \mathbb{T}^m \times E$, where E is a finite group and $\mathbb{T}^m$ is the direct product of m-copies of the solenoid group $\mathbb{T}$. On another hand, $N(G)$ has each proper quotient group which is a compact abelian Lie group. Then it satisfies [10, Theorem 2.1] and so $N(G) \simeq \mathbb{Z}_p$ for some prime p. The result follows. □

3. Some examples

In this section we illustrate some constructions correlated to the last results.

Example 3.1. Let A be an arbitrary commutative (Hausdorff) topological group and consider $G = A \rtimes \{1, -1\}$ with multiplication $(m, \epsilon)(m', \epsilon') = (m + \epsilon m', \epsilon\epsilon')$, where $m, m' \in A$ and $\epsilon, \epsilon' \in \{1, -1\}$. Of course, G is a metabelian group. Some authors call G the *general dihedral group*. G is compact if and only if A is compact. If C is any closed subgroup of A, $N = C \times \{1\}$ is a closed normal subgroup and $G/N \simeq (A/C) \rtimes \{-1, 1\}$ is again a general dihedral group which is metabelian (non-abelian), if $C \neq A$.

Case 1: non-discrete group. Assume $A = \mathbb{T}$, the solenoid group. The group $G = \mathbb{T} \rtimes \{1, -1\}$ is a finite extension of a compact group so that it is still a compact group. We have $Z(G) = (\frac{1}{2}\mathbb{Z}/\mathbb{Z}) \times \{1\}$ and the hypotheses of Theorem 2.10 are fulfilled, considering Ω as the prevariety of finite groups. In particular, G is metabelian non-centerfree and non-discrete.
By taking the direct product of finitely many finite cyclic groups and G, we may construct a solvable JNΩ group of arbitrary derived length.

Case 2: discrete group. Assume $A = \mathbb{Q}_p$ is the group of p-adic rationals for some prime p. G is a Baumslag-Solitar group (see [8, Chapter 11]). In particular, G is discrete and non-compact. Then Theorem 2.10 cannot be applied as it stands. Note that A is the unique minimal normal subgroup of G. Furthermore, $N(G) = Z(G) = \{1\}$. If Ω is the prevariety of finite groups, then it is easy to check that each proper quotient of G is finite. From another hand, G is a non-compact $JN\Omega$ group. □

Remark 3.2. *Case 2 of Example 3.1 shows the importance of the compactness in Theorem 2.10. Moreover it suggests that we may still improve Theorem 2.10, trying to consider more possibilities, for instance weakening the hypothesis.*

We further remark that for the group of case 1 we cannot speak about its behavior at infinity since it is not discrete. Whereas in case 2, since G is discrete and finitely presented, one can look at its asymptotic properties as in [9, 3]. In particular one sees that that G is quasi-simple filtered from [3, Corollary 4.1 (3)]) but not simply connected at infinity from [3, §4.2]. These properties are topological tameness conditions of the Cayley 2-complex associated to a presentation of the group (see [9, 3]). Hence one might wonder whether there are connections between the theory of JNΩ-groups and the asymptotic geometry of such groups (in the discrete case).

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