GENERALIZED HYPERCENTERS IN INFINITE GROUPS

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ABSTRACT. We consider the so-called generalized center, defined by Agrawal, in the slightly wider context of periodic groups and try to find out where additional conditions are needed for refinements. In particular we consider the final terms of the corresponding ascending sequences.

1. Generalized hypercenters

We will consider only periodic groups. The widely known concept of a center $Z(G)$ of a group G has been subjected to generalizations: it was extended to the subgroup $Q(G)$ generated by all cyclic normal subgroups and to the subgroup generated by all cyclic quasinormal subgroups (see [4, 5, 9, 10, 11, 12]). The next generalization refers to a still more general setting.

A subgroup H of a finite group G is called *S–permutable* if it permutes with all Sylow subgroups of G. It was shown by Kegel [7] that S–permutable subgroups are subnormal. Agrawal [1, 12] introduced the *generalized center*

$$genz(G) = \langle x \mid \langle x \rangle \text{ is S} - \text{permutable in } G\rangle. \tag{1.1}$$

Sylow subgroups need not exist in infinite groups. Here we define an element x to be *M–permutable* if $\langle x \rangle$ is permutable with all maximal p–subgroups of G for all primes p and introduce the generalized center

$$gz(G) = \langle x \mid \langle x \rangle \text{ is M} - \text{permutable in } G\rangle. \tag{1.2}$$

We form ascending series for all these concepts. Let $\zeta(G)$ be one of these concepts, we define the sequence

$$1 = \zeta_0(G) \subseteq \zeta_1(G) = \zeta(G) \subseteq \zeta_2(G) \subseteq \dots, \tag{1.3}$$

with

$$\zeta_{\alpha+1}(G)/\zeta_\alpha(G) = \zeta(G/\zeta_\alpha(G)) \tag{1.4}$$

for all ordinal numbers α, and for limit ordinals λ

$$\zeta_\lambda(G) = \bigcup_{\sigma<\lambda} \zeta_\sigma(G). \tag{1.5}$$

The end of the sequence is reached when $\zeta_\beta(G) = \zeta_{\beta+1}(G)$, and we denote this last term by $\zeta_\infty(G)$.

Asaad and Mohamed [3, Definition, p.2240] define a normal subgroup H of a finite group G to be *generalized supersolvably embedded, (GSE)* in G if there is a chain of subgroups

$$1 = H_0 \subseteq H_1 \subseteq H_2 .... \subseteq H_\lambda = H \tag{1.6}$$

such that H_α is S–permutable in G and all $|H_{\alpha+1} : H_\alpha|$ are primes. They show that the product of all GSE–subgroups of G coincides with $genz_\infty(G)$. We extend this to infinite groups, on one hand we allow infinite sequences with the proviso of (1.3), the maximal term we call $GSE(G)$, on the other hand we use the same concept for M–permutable subgroups, the maximal term we denote by $GZE(G)$. We will compare the different final terms.

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2. M–permutable elements

M–permutable elements are described completely by the following lemma.

Lemma 2.1. *The following three statements hold in a periodic group G:*

- **(i):** *Let p be a prime and x be of order a power of p. Then x is M–permutable in G, if and only if it is contained in the maximal normal p–subgroup of G and $\langle x\rangle$ is normalized by all p'–elements.*
- **(ii):** *If x is M–permutable in G, then all powers of x are M–permutable.*
- **(iii):** *If x and y are M–permutable and $(o(x), o(y)) = 1$, then xy is M–permutable.*

Proof. In all 3 cases, x is an element of finite order, so if $\langle x\rangle H = H\langle x\rangle = K$ for some subgroup H, then the normal core H_K of H in K is of finite index in K and we may use the Sylow theorems in K/H. In all arguments below we put $\langle x\rangle = X$.

(i). Let P be a maximal p–subgroup of G and assume that X is M–permutable. Then $XP = PX$ is a p–subgroup, by maximality of P coinciding with P. So x belongs to all maximal p–subgroups of G, and they intersect in the maximal normal p–subgroup N_p of G. Let $q \neq p$ be a prime and Q a maximal q–subgroup of G. We have $N_p \cap XQ = X$, so X is normalized by all maximal q–subgroups and so by all q–subgroups. The other direction is trivial.

(ii). Assume now that x has any finite order and is M–permutable. Denote the Sylow p–subgroup of x by $Y = \langle y\rangle$. If P is a maximal p–subgroup of G, then P is also a maximal p–subgroup in PX and so $Y \subseteq P$. If $q \neq p$ and Q is a maximal q–subgroup of G, then the maximal p–subgroups of QX are conjugates of Y, and all of them belong to N_p. So Y is normalized by Q and by all q–subgroups of G. The same applies for all subgroups of Y. Now $X = Y \times C$ where C is the Hall p'–subgroup of X. By an obvious induction on $o(x)$ we obtain (ii).

(iii). We see that x and y are contained in normal subgroups intersecting trivially, so $xy = yx$ and $XY = YX$. If M is a maximal p–subgroup of G, then $XYM = XMY = MXY$. The proof is complete. □

We now compare the subgroups $GZE(G)$ and $gz_\infty(G)$.

Lemma 2.2. *Let G be a group and H be a normal subgroup of G properly contained in $gz_\infty(G)$. Then the following statements are true.*

- **(i):** $GZE(G) \leq gz_\infty(G)$.
- **(ii):** *$gz(G/H)$ is non–trivial.*
- **(iii):** $gz_\infty(G/H) = gz_\infty(G)/H$.
- **(iv):** *If, for all primes p, all p–subgroups of quotient groups G/H of G satisfy the normalizer condition, then $gz_\infty(G)$ is generalized supersolvably embedded in G.*

Proof. (i). Suppose that the result is wrong. Let K be a generalized supersolvably embedded subgroup of G. Then there is a maximal subgroup H_α in the series given for K, such that $H_\alpha \leq gz_\infty(G)$, in particular, $H_{\alpha+1}$ is not a subgroup of $gz_\infty(G)$. Now $gz_\infty(G)H_{\alpha+1}$ is M–permutable and $gz_\infty(G)H_{\alpha+1}/gz_\infty(G) \subseteq gz(G/gz_\infty(G)) = 1$, a contradiction.

(ii). Let λ be an ordinal such that $gz_\lambda(G) \leq H$ and $gz_{\lambda+1}(G) \not\leq H$. Hence there is an element $y \in gz_{\lambda+1}(G) \setminus H$ such that $ygz_\lambda(G)$ is M–permutable in $G/gz_\lambda(G)$. Then yH is a non–trivial element of $gz(G/H)$ as claimed.

(iii). Since an epimorphic image of an M–permutable subgroup is again M–permutable, $gz_1(G)H/H \leq gz_1(G/H)$. From this it is easy to see that $gz_2(G)H/H \leq gz_2(G/H)$. Let λ be an ordinal and assume that $gz_\alpha(G)H/H \leq gz_\alpha(G/H)$ for all ordinals $\alpha < \lambda$. If λ is not a limit ordinal, then $\lambda - 1$ exists and it is clear that $gz_\lambda(G)H/H \leq gz_\lambda(G/H)$. If λ is a limit ordinal, then it is straightforward to show that $gz_\lambda(G)H/H \leq gz_\lambda(G/H)$. Thus $gz_\infty(G)H/H \leq gz_\infty(G/H)$. Assume now that $gz_\infty(G)H/H$ is properly contained in $gz_\infty(G/H)$. By (ii) $gz((G/H)/gz_\infty(G)/H)$ is non-trivial. Hence the same is true for $G/gz_\infty(G)$. This is a contradiction and (iii) follows.

(iv). We construct sequences of subgroups as in (1.6)

$$1 = H_0 \subset H_1 \subset H_2 \ldots \subset H_\alpha \subset \ldots \tag{2.1}$$

with the following conditions:

- **(1):** $|H_{\alpha+1} : H_\alpha|$ is a prime for all ordinals α,
- **(2):** for limit ordinals λ we define $H_\lambda = \cup_{i<\lambda} H_\alpha$,
- **(3):** for every ordinal α there is an ordinal β such that $gz_\beta(G) \subseteq H_\alpha \subset gz_{\beta+1}(G)$ and $H_\alpha/gz_\beta(G)$ is generated by M–permutable elements of $G/gz_\beta(G)$.

All H_α are M–permutable by (3), and the maximal term $B = \bigcup H_\alpha$ is generalized supersolvably embedded. We consider the set of all such sequences. This is an ordered set Σ, the sequence $\mathcal{S}_1 < \mathcal{S}_2$ whenever $\mathcal{S}_2$ contains all terms of $\mathcal{S}_1$ and at least one more. We find that with every ordered subset of Σ also the set-theoretic union of all members belongs to Σ. This means that we may apply the Maximum Principle of Set Theory: the set Σ contains maximal elements. Let $\mathcal{S}$ be such a maximal element in Σ and put $B = \bigcup_{H_\alpha \in \mathcal{S}} H_\alpha$. If $B = gz_\infty(G)$ we are finished. If not, we have by (3) an ordinal γ such that $gz_\gamma(G) \subseteq B \subset gz_{\gamma+1}(G)$. We have two cases: $B/gz_\gamma(G)$ is a normal subgroup of $gz_{\gamma+1}(G)/gz_\gamma(G)$ or not. In the first case we choose an element $xgz_\gamma(G) \in gz_{\gamma+1}(G)/gz_\gamma(G)$ which is not contained in $B/gz_\gamma(G)$ but $x^p gz_\gamma(G)$ is for some prime p, in addition it is M–permutable. Now also $\langle x\rangle B$ is generalized supersolvably embedded and $\mathcal{S}$ is not maximal in Σ, a contradiction. For the second case we need the normalizer condition. Since $gz_\gamma(G) \subseteq B \subset gz_{\gamma+1}(G)$ and $gz_{\gamma+1}(G)/gz_\gamma(G)$ is a direct product of p–subgroups we have that $gz_{\gamma+1}(G)/gz_\gamma(G)$ satisfies the normalizer condition and

$$B \neq N(B) \cap gz_{\gamma+1}(G) \neq gz_{\gamma+1}(G). \tag{2.2}$$

Now again

$$N(B) \cap gz_{\gamma+1}(G) \neq N(N(B) \cap gz_{\gamma+1}(G)) \cap gz_{\gamma+1}(G). \tag{2.3}$$

We choose $w \in N(N(B) \cap gz_{\gamma+1}(G)) \cap gz_{\gamma+1}(G) \setminus N(B) \cap gz_{\gamma+1}(G)$. By construction, $w^{-1}Bw \neq B$ and $B/gz_\gamma(G)$ is generated by M–permutable elements of $G/gz_k(G)$. So there is an M–permutable element $ugz_\gamma(G) \in B/gz_\gamma(G)$ such that $w^{-1}uwgz_\gamma(G) \notin B/gz_\gamma(G)$ but $w^{-1}u^p w \in B$ for some prime p and also $w^{-1}uw \in N(B)$ is true. We obtain with the existence of $\langle w^{-1}uw\rangle B$ that $\mathcal{S}$ is not maximal in Σ, the final contradiction showing that $B = gz_\infty(G)$, which means that (iv) is true.

□

The importance of Lemma 2.2 is due to the fact that we have several obstacles comparing (1.1)–(1.6). We indicate this by two examples for the convenience of the reader.

Example 2.3. Let X be a Tarski group for the prime p, that is, X is an infinite p–group with all proper subgroups of order p. Then $gz(X) = genz(X) = X$ while $GZE(X) = GSE(X) = 1$ since the other normal subgroup X is not generalized supersolvably embedded. So there are no generalized supersolvably embedded subgroups in X except 1. Indirectly we have also shown that the normalizer condition is really necessary in Lemma 2.2 (iv) but not in remaining statements (i)–(iii).

Example 2.4. Let $W = \langle a\rangle \wr (\langle b\rangle \times E_p)$, where $|\langle a\rangle| = p$, an odd prime, $|\langle b\rangle| = 2$, and E_p is an infinite elementary abelian p–group. Then, for every x in E_p, $\langle a, b, x\rangle^W$ is generalized supersolvably embedded and also W is generalized supersolvably embedded. We have $genz(W) = gz(W) = GZE(W) = GSE(W) = W$. However, the maximal p–subgroup W^2 possesses selfnormalizing proper subgroups, for instance E_p. We could not decide whether local finiteness is sufficient in Lemma 2.2.

From [6, p. 28], we recall that $O^p(G) = \langle x \mid (o(x), p) = 1\rangle$ and $O_p(G) = \langle N \mid N \triangleleft G; N \text{ is a } p-\text{group}\rangle$. The group of the automorphisms induced by G on $gz_\infty(G)$ is described below.

Theorem 2.5. *Let G be a group.*

- **(i):** *$O^p(G) \cap O_p(gz_\infty(G))$ is hypercentral.*

(ii): *Let $M(p) = O_p(gz_\infty(G))$. There are normal subgroups $L(p), K(p)$ of G such that*

$$C_G(M(p)) \leq L(p) \leq K(p) \leq G \tag{2.4}$$

where $L(p)/C_G(M(p))$ and $G/K(p)$ are p–groups while $K(p)/L(p)$ is abelian of exponent dividing $p-1$.

Proof. (i). Let H be an epimorphic image of G. Let $x \in O^p(H)$ be of order p^m and an M–permutable element of H. Then $\langle x\rangle \triangleleft O^p(H)$ and $O^p(H)/C_G(x) \cap O^p(H)$ is cyclic of order dividing $p-1$. So we have $x \in Z(O_p(H) \cap O_p(gz_\infty(H))$ and $O^p(H) \cap O_p(gz(H)) \subseteq Z(O^p(H) \cap O_p(gz_\infty(H)))$. Now (i) follows.

(ii). Put $K(p) = C_G(M(p))O^p(G)$ and $L(p) = C_G(M(p))K(p)'K(p)^{p-1}$. We deduce from (i) that $[L(p), O_p(gz_{\alpha+1}(G)) \cap O^p(G)] \subseteq gz_\alpha(G)$ for all ordinal numbers α. So for all $y \in L(p)$ and $x \in O^p(G) \cap O_p(gz_\infty(G)$ the sequence $[y, x], [y, [y, x]], [y, [y, [y, x]]], ...$ ends with 1. Since G is a torsion group, y must be of p–power order. Now $L(p)/C_G(M(p))$ and $G/K(p)$ are p–groups, and (ii) is true. □

Theorem 2.5 is trivially true for torsion–free groups and so it is significant only in the periodic case. Note that no restrictions on G are assumed in Theorem 2.5. Another result of general interest is the following.

Lemma 2.6. *Let G be a group.*

(i): *If N is a normal subgroup of G that is contained in $gz(G)$, then N is generated by M–permutable elements.*

(ii): *If N is a normal subgroup of G and $N \cap gz_\infty(G) \neq 1$, then $N \cap gz(G) \neq 1$.*

Proof. (i). Assume first that N is a p–group. If $N \cap O^p(G) = 1$, then all elements of N are M–permutable. Let now $K = N \cap O^p(G) \neq 1$ and choose an element $x \in K$ of order a power of p. Since $x \in gz(G)$ it is a product $y_1 y_2 ... y_k$ of M–permutable elements and so the smallest normal subgroup L of $O^p(G)$ that contains x is contained in $\langle y_1, \ldots, y_k\rangle$. It follows that L is finite and $O^p(G)/(O^p(G) \cap C_G(L))$ is abelian of exponent dividing $p-1$ since $C_G(L) \leq \bigcup_{i=1}^{k} C_G(y_i)$. This shows that L is a product of cyclic $O^p(G)$–invariant subgroups, their generators are M–permutable. So K is generated by M–permutable elements. The same is true for N since $N = [N, O^p(G)] \times N \cap C(O^p(G))$ and $NO^p(G) = \langle N, O^p(G)\rangle \leq K$. This shows (i) for normal p–subgroups. Now assume N is any normal subgroup of G contained in $gz(G)$. Then N is a product of normal p–subgroups and so (i) is true in general.

(ii). We assume $N \neq 1$ and we denote by α the maximal ordinal number such that $gz_\alpha(G) \cap N = 1$. Let $gz_\alpha(G) = A$ and $gz_{\alpha+1}(G) = B$. Then $B \cap N \neq 1$. Since $B/A = gz(G/A)$ and $(NA/A) \cap (B/A) \neq 1$, there is an element $yA \in (NA/A) \cap (B/A)$ which is M–permutable in G/A. But then $y = N \cap yA$ is M–permutable, $y \in gz(G)$ and $A = 1$. This shows (ii). □

In case of hypercentral groups we have relations among (1.3) and the upper central series. More details are offered by the next result.

Proposition 2.7. *Let D satisfy the following conditions:*

(i): *$D \triangleleft gz_\infty(G)$ is a direct product of its maximal primary subgroups.*

(ii): *D is maximal with respect to (i) above.*

Then $gz(G/D) = gz_\infty(G)/D = Z(G/D)$.

Proof. Let $uD \in G/D$ be of order p^m and an M–permutable element of G/D. By maximality of D we have $\langle uD\rangle C(O^p(D))/D = 1$. So there is a prime q such that $\langle uD\rangle \cap C(O_q(D))/D = 1$ and p^m must be a divisor of $q-1$. By Theorem 2.5 (ii) we obtain that $(G/D)/C_{G/D}(uD)$ is cyclic of order a power of q, on the other hand, $(G/D)/C_{G/D}(uD)$ is cyclic of order dividing $p-1$ since it is of order p^m. This shows $C_{G/D}(uD) = G/D$ and $uD \in Z(G/D)$. We see $gz(G/D) \subseteq Z(G/D)$, the other inclusion $Z(G/D) \subseteq gz(G/D)$ is trivial. Consider an M–permutable element $vZ(G/D)$

of $G/D/Z(G/D)$, assume that it is of order p. Then $\langle v, Z(G/D)$ is abelian and normalized by $O^p(G/D)$. If there is no M–permutable element wD of G/D with $wZ(G/D) = vZ(G/D)$, then there is an element t such that $[v,t]D \in Z(G/D)$ and $[v,t]D = uD \neq D$ for some u as before. We may assume that t is of order a power of p and $t \notin C_G(O_q(D))$. Maximal p–subgroups of $G/C_G(O_q(D))$ are abelian by Theorem 2.5 (ii), and so t does not exist. So $gz(G/D) = gz_2(G/D)$. □

Example 2.8. Let T be a Tarski group for the prime p, i. e. a simple infinite p - group with all proper subgroups of order p and let E be the non-split extension of T by a group U of order 2; denote the group of order p by C. If $G = C \wr E$ with $B = C^E$, then $gz_\infty(G) = gz_2(G) = BU$. In particular, $BU \subseteq BG' = G$.

Example 2.9. Let H be a hypercentral p–group, where p is a odd prime. Assume that there is an extension $K = \langle x, H\rangle$ with $x^2 = 1$ and $K' = H$. If L is a finite p–group, then $gz_\infty(K \wr L) = \langle H^L, y\rangle$ where $y \neq 1$ and $y \in Z(\langle x\rangle \wr L)$. (If L is infinite, y does not exist).

3. Hyperfinite groups

Let $\mathcal{P}$ be a property which is defined within classes of groups. We say that a group G is a hyper–$\mathcal{P}$–group if for every proper normal subgroup $N \neq G$ of G, there is a normal subgroup $M/N \neq 1$ such that M/N has property $\mathcal{P}$ in G. In particular, we are interested in the present section to the case in which $\mathcal{P}$ stands for "central", "finite" and "cyclic". Recent contributions can be found in [2], where more general situations are considered. As usual, the *Hirsch-Plotkin radical* of a group is its maximal locally nilpotent normal subgroup. We want to exploit how much our statement can be strengthened in hyper–$\mathcal{P}$–groups. Recall that Kovacs, Neumann and de Vries [8] have constructed (among others) countable hyperfinite groups of exponent 6 with non-conjugate maximal 2–subgroups. Modifying this to exponent 20, maximal 2–subgroups may be non-isomorphic.

Theorem 3.1. *If G is a hyperfinite group, then*

- **(i):** *$gz_\infty(G)$ is hypercyclic,*
- **(ii):** *$gz_\infty(G)G'/HP(gz_\infty(G))G'$ is complemented in G/DG'.*

Proof. (i). It suffices to show that $gz(G)$ is hypercyclically embedded in $gz_\infty(G)$. Let $L = O_p(gz(G)) \neq 1$ and consider a finite normal subgroup $M \subseteq L$. If $M \cap O^p(G) \neq 1$, then we have a cyclic normal subgroup of $gz_\infty(G)$ in M by Lemma 2.6 (i). If $M \cap O^p(G) = 1$, then $G/C_G(M)$ is a finite p–group and $Z(G) \cap M \neq 1$. So (i) is shown.

(ii). By Proposition 2.7 (ii), $gz_\infty(G)/D = Z(G/D)$. Again it suffices to consider the maximal p–subgroup R/D of $gz_\infty(G)/D$. Let p divide $q-1$ where q is some prime, and assume that $R \not\subseteq C_G(O_q(D))$. Using the notation of the proof of Theorem 2.5 (ii), $RL(q)/L(q) \subseteq Z(G/L(q))$, also $G/K(q)$ is a q–group while $RL(q)/L(q)$ is contained in the normal subgroup $K(q)/L(q)$ which is abelian of exponent dividing $q-1$. We have $\langle (RL(q)/L(q)), (O^p(G)/L(q))\rangle = G/L(q)$ and $\big(RL(q)/L(q)\big) \cap \big(O^p(G/L(q))\big) = 1$ since $G/L(q)$ is locally finite. Put $S(q) = O^p(G/L(q))$. The complement of R/D is now $\bigcap_{q\neq p} S(q)$, obviously this contains DG'. □

Now Theorem 3.1 leads to the following.

Theorem 3.2. *If M is a hypercyclic subgroup of the hyperfinite group G, then $Mgz_\infty(G)$ is also hypercyclic.*

Proof. Denote $Mgz_\infty(G)$ and $gz_\infty(G)$ by U, V respectively. We consider $U \cap C_G(D)$ where D is the Hirsch-Plotkin radical of V: we want to show first that D is hypercyclically embedded in $Mgz_\infty(G)$. Let $H, K \subseteq D$ be normal subgroups of U such that H/K is a minimal normal subgroup of U/K. Then H/K is elementary abelian of exponent, say, p. The quotient group $(G/K)/C_{G/K}(H/K)$ is in turn isomorphic to a quotient group of $G/C_G(O_p(D))$, its structure is given in Theorem 2.5 (ii):

it has a normal series with a p–group, an abelian group of exponent dividing $p-1$, and a p–group. Likewise

$$(U/K)/(U/K \cap C_{G/K}(H/K)) \cong (U/K)C_{G/K}(H/K)/C_{G/K}(H/K) \tag{3.1}$$

is a subgroup of $(G/K)/(H/K)$. Since H/K is a p–group and a minimal normal subgroup of U/K, the normal p–subgroup of $(U/K)/((U/K) \cap C_{G/K}(H/K))$ is trivial, and this quotient is an extension of an abelian group of exponent dividing $p-1$ by a p–group, further

$$(V/K)/(D/K) \cong V/U \subseteq Z(G/D) \tag{3.2}$$

is abelian. Now $L(p)$ in the sense of Theorem 2.5 (ii) satisfies $L(p) = D(C_G(O_p(D))$ and

$$(V/K)C_{G/K}(H/K)/C_{G/K}(H/K) \subseteq Z((U/K)C_{G/K}(H/K)/C_{G/K}(H/K). \tag{3.3}$$

On the other hand, $(MK/K)/\big((MK/K \cap C_{G/K}(H/K))\big)$ is hypercyclic and a subgroup of a quotient group of $G/C_G(O_p(D))$. Since $U = VM$ we obtain that $(U/K)/C_{G/K}(H/K)$ is a product of a subgroup

$$(V/K)C_{G/K}(H/K)/C_{G/K}(H/K) \cong (V/K)/((V/K) \cap C_{G/K}(H/K), \tag{3.4}$$

a subgroup of the center, and of $(MK/K)C_{G/K}(H/K)/C_{G/K}(H/K)$, a hypercyclic group. Now $(V/K)/C_{G/K}(H/K)$ is hypercyclic such that its normal p - subgroups are trivial. Applying Theorem 2.5 (ii) again we have that $(V/K)/C_{G/K}(V/K)$ is abelian of exponent dividing $p-1$. So H/K must be cyclic, and D is hypercyclically embedded in U. On the other hand, $U/D = (V/D)(MD/D)$ and $MD/D \cong M/M \cap D$ is hypercyclic. Using Theorem 3.1 (i) we find that U/D is hypercyclic and the result follows. □

A direct consequence of Theorem 3.2 is the following.

Corollary 3.3. *Let G be a finite group. Then $genz_\infty(G)$ is contained in the intersection of all maximal supersolvable subgroups of G.*

We note that Corollary 3.3 implies that the maximal supersolvably embedded normal subgroups of a finite group G may be a proper subgroup of the intersection of all maximal supersolvable subgroups of G. For locally finite groups we obtain the equality under certain finiteness conditions.

Corollary 3.4. *Let G be a locally finite group. Assume that the primary subgroups of G satisfy the minimum condition and the normalizer condition. Then*

$$GSE(G) = GZE(G) = gz_\infty(G) = genz_\infty(G) \tag{3.5}$$

is a hypercyclic group.

Proof. By Lemma 2.2 (iv) $GSE(G) = gz_\infty(G)$. On the other hand, the S–permutable subgroups in a locally finite group with *min-p* for all primes p are exactly the M–permutable subgroups. Therefore $gz_\infty(G) = genz_\infty(G)$ and $GSE(G) = GZE(G)$. We conclude that $GSE(G) = GZE(G) = gz_\infty(G) = genz_\infty(G)$. In particular we may refine the series (1.3) to obtain a series with G–invariant terms and cyclic factors of prime order. This means $gz_\infty(G)$ is hypercyclic. This completes the proof. □

Comparing [1, 3, 4, 5, 9, 10, 11, 12] we note that $genz(G)$ is introduced in order to generalize the *quasicenter* $Q(G)$ of a group G. Recall that $Q(G)$ is defined to be the subgroup of G generated by all elements g of G such that $\langle g\rangle$ is permutable with all the subgroups of G. Of course, $Q(G) \subseteq genz(G) \subseteq gz(G)$ and we may easily specialize (1.3)–(1.5), getting the *hyperquasicenter* $Q_\infty(G)$ of G. This was introduced by Mukherjee in [9, 10] and studied in [5]. Of course, $Q_\infty(G) \subseteq genz_\infty(G) \subseteq gz_\infty(G)$. The next result improves strongly [11, Main Theorem].

Corollary 3.5. *If G is a hyperfinite group in which $Q_\infty(G)$ is finite, then $G/C_G(Q_\infty(G))$ is finite and supersolvable. In particular, if $Q_\infty(G)$ is a p–group, then $G/C_G(Q_\infty(G))$ is an extension of a p–group by an abelian group of exponent dividing $p-1$.*

Proof. Let $\sigma(G) = Q_\infty(G)$ be the maximal hypercyclically embedded normal subgroup of G (see [5, Theorem 1]). Since $\sigma(G) = Q_\infty(G)$ is finite, we may proceed by induction on $|\sigma(G)|$. As a first step we show:

(i). If N is a finite normal p–subgroup of G and N is hypercyclically embedded in G, then $G/C_G(N)$ is supersolvable.

Consider a chief series $1 = P_0 \triangleleft P_1 \triangleleft \ldots \triangleleft P_k = N \triangleleft \ldots$ of G, beginning with 1 and containing N $(k \geq 1)$. Then $G'G^{p-1}$ stabilizes this part of the chief series, and $G'G^{p-1}C_G(N)/C_G(N)$ is a p–group. This means that $G/C_G(N)$ is an extension of a p–group by an abelian group of exponent dividing $p-1$; it is also finite since N is finite. Therefore $G/C_G(N)$ is supersolvable. We conclude that the result is completely proved in case $\sigma(G)$ is a p–group.

Assume now that $\sigma(G)$ is not a p–group and that the theorem is true for every subgroup K of G such that $|\sigma(G)| > |\sigma(K)|$. Let q be the biggest prime divisor of $|\sigma(G)|$. There is a normal subgroup W of G such that $|W| = q$, and this is contained in $\sigma(G)$. Denote the Sylow q–subgroup of $\sigma(G)$ by Q; this is a characteristic subgroup of $\sigma(G)$ and therefore normal in G. By (i), $G/C_G(Q)$ is supersolvable.

Let $C_{G/W}(\sigma(G/W)) = V/W$. By induction, $(G/W)/(V/W) \cong G/V$ is supersolvable, and, since the class of supersolvable groups is a formation:

(ii). $G/(V \cap C_G(Q))$ is supersolvable.

Let $y \in (V \cap C_G(Q)) \setminus C_G(\sigma(G))$. Then there are $x \in \sigma(G)$ and $z \in W$ such that $y^{-1}xy = xz$; the element x is not contained in Q since $x \notin V$. If $W \subseteq Z(\sigma(G))$, then x and xz have different order, a contradiction. We deduce that $W \cap C_G(\sigma(G)) = 1$ and $x \in \sigma(G) \setminus C_{\sigma(G)}(W)$.

Now $\sigma(G)/C_{\sigma(G)}(W)$ is cyclic of order dividing $q-1$ since $|W| = q$. Let s be of smallest possible order such that $\sigma(G) = \langle s, C_{\sigma(G)}(W)\rangle$. We have $y^{-1}sy = sz$ for some $z \in W$. But also $s^{-1}zs = z^k$ for some k. We have $s^{-1}y^{-1}sy = z$ and $s^{-1}zsz^{-1} = z^{k-1}$. Take m such that $m(k-1) \equiv 1$ mod. q. Then $y^m z \in C_G(s)$. We obtain therefore $V \cap C_G(Q) = C_G(\sigma(G))$, and the proof is complete by (ii). $\square$

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