Recent Trends in Probability in Group Theory in Some Classes of Compact Groups

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Abstract. The topic of the present work is related to a new branch of Statistical Sciences which is arousing interest in the last twenty years: the Probability in Group Theory. In the finite groups the probability that two randomly chosen elements commute has been widely investigated by many authors. There are some classical results which estimate the bounds for this kind of probability so that the knowledge of the whole structure of the group can be more accurate.

The same topic has been recently extended to certain classes of infinite compact groups, obtaining restrictions on the group of the inner automorphisms. Here such restrictions are improved for a wider class of infinite compact groups.

Key words: Compact groups; probability in group theory.

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1 Introduction and Statement of Results

Let G be a compact, Hausdorff topological group. We recall that G has a left Haar measure, that is, a Borel measure μ_G such that $\mu_G(U) > 0$ for each nonempty open set U of G, and $\mu_G(x \cdot E) = \mu_G(E)$ for each Borel set E of G and each $x \in G$. Further, μ_G is unique once we impose the normalization condition $\mu_G(G) = 1$. See for details (Hewitt & Ross, 1963, Sections 18.1, 18.2, Proposition 18.2.1) or (Hofmann & Morris, 1998, Theorem 2.8). On the product space $G \times G$, we consider the product measure $\mu_G \times \mu_G$. If

$$C_2 = \{(x, y) \in G \times G \mid [x, y] = 1\},$$

then $C_2 = f^{-1}(1)$, where $f : (x, y) \in G \times G \mapsto f(x, y) = [x, y] = x^{-1}y^{-1}xy \in G$ and 1 denotes the neutral element of G. It is clear that f is continuous and C_2

is a compact, and hence measurable, subset of $G \times G$. This allows us to define the *commutativity degree*

$$d(G) = (\mu_G \times \mu_G)(C_2)$$

of G. The reader may find a similar treatment in the work (Gustafson, 1973, Section 2) of Gustafson, who introduced $d(G)$ around 40 years ago. Now the concept of $d(G)$ was subject of generalizations as follows. Suppose that $n \geq 1$ and G^n is the product of n-copies of G and $\mu_G^n = \underbrace{\mu_G \times \mu_G \times \ldots \times \mu_G}_{n-times}$. Then we may consider the *n-th commutativity degree*

$$d^{(n)}(G) = \mu_G^{n+1}(C_{n+1})$$

of G, where

$$C_{n+1} = \{(x_1, \ldots, x_{n+1}) \in G^{n+1} \ |[x_1, x_2, ..., x_{n+1}] = 1\}.$$

Obviously, if G is finite, then G is a compact group with the discrete topology and so the Haar measure of G is the counting measure. Then we have as special situation for a finite group G:

$$d^{(n)}(G) = \mu_G^{n+1}(C_{n+1}) = \frac{|C_{n+1}|}{|G|^{n+1}}.$$

More generally, let H be a closed subgroup of a compact group G. It is possible to define

$$D_2 = \{(h, g) \in H \times G \mid [h, g] = 1\},$$

then $D_2 = \phi^{-1}(1)$, where $\phi : (h, g) \in H \times G \to \phi(h, g) = [h, g] \in G$. It is clear that ϕ is continuous and so D_2 is a compact and measurable subset of $H \times G$. Note that ϕ is the restriction of f to $H \times G$. This remark shows why H has to be required as closed subgroup of G. Then we may define the *relative commutativity degree* of H with respect to G as

$$d(H, G) = (\mu_H \times \mu_G)(D_2).$$

Considering

$$D_{n+1} = \{(h_1, , ..., h_n, g) \in H^n \times G \ |[h_1, h_2, ..., h_n, g] = 1\},$$

we may define at the same way the *relative n-th commutativity degree* of H with respect to G as

$$d^{(n)}(H, G) = (\mu_H^n \times \mu_G)(D_{n+1}).$$

The main result of the present paper is listed below in case that $n = 1$. As usual, $Z(G) = \{a \in G \mid [a, b] = 1, \ b \in G\}$ denotes the center of G.

Main Theorem. *Let H be a closed subgroup of a compact group G.*

(i) *If $d(H, G) = \frac{3}{4}$, then $H/(Z(G) \cap H)$ is cyclic of order 2.*

(ii) *If $d(H, G) = \frac{5}{8}$ and H is nonabelian, then $H/(Z(G) \cap H)$ is a 2-elementary abelian group of rank 2.*

We end with a terminological observation. It is known that there are different ways to denote the direct product of 2 cyclic groups of order 2: For instance, some authors denote it with $Z_2 \times Z_2$, or $\mathbb{Z}_2 \times \mathbb{Z}_2$, or $\mathbb{Z}(2) \times \mathbb{Z}(2)$, or $C_2 \times C_2$. In the present note, we call it a *2-elementary abelian group of rank 2*, in order to avoid superfluous symbols.

2 Proof of Main Theorem

This section is devoted to prove Main Theorem

Lemma A. *Assume that G is a compact group, H is a closed subgroup of G and h belongs to H. Then*

$$d(H,G) = \int_H \mu_G(C_G(h)) d\mu_H(h),$$

where $C_G(h) = \{g \in G \mid [g,h] = 1\}$ is the centralizer of h in G,

$$\mu_G(C_G(h)) = \int_G \chi_{D_2}(h,g) d\mu_G(g),$$

g belongs to G and χ_{D_2} denotes the characteristic map of the set D_2.

Proof. Since

$$\mu_G(C_G(h)) = \int_G \chi_{D_2}(h,g) d\mu_G(g),$$

we have by Fubini-Tonelli's Theorem:

$$d(H,G) = (\mu_H \times \mu_G)(D_2) = \int_{H \times G} \chi_{D_2}(d\mu_H \times d\mu_G)$$

$$= \int_H \int_G \chi_{D_2}(h,g) d\mu_G(g) d\mu_H(h) = \int_H \mu_G(C_G(h)) d\mu_H(h).\square$$

Proof of Main Theorem. (i). Assume that $d(H,G) = \frac{3}{4}$ and $K = H \cap Z(G)$. Since μ_G is monotone and normalized, $\mu_G(C_G(h)) \leq \mu_G(G) = 1$, where h belongs to K. Now, let h be an element of H not belonging to K. We put $C_G(h) = L$ and note that it is a closed subgroup of G. Incidentally, we recall that the centralizer of a closed subgroup is again a closed subgroup by a trivial argument, which can be found in (Hewitt & Ross, 1963) or (Hofmann & Morris, 1998). We have the following generalization to groups with measures of Lagrange's Theorem on the index of subgroups

$$1 = \mu_G(G) = |G : L| \; \mu_G(L),$$

where $|G : L|$ denotes the index of L in G. Details can be found in (Hewitt, E. & K. A. Ross, 1963). From this fact, if $|G : L| \geq 2$, then $\mu_G(L) \leq \frac{1}{2}$. But, the choice of h implies that L is a proper subgroup of G, so $|G : L| \neq 1$. Therefore, $|G : L| \geq 2$ and so $\mu_G(L) \leq \frac{1}{2}$. This means that $\mu_G(C_G(h)) \leq \frac{1}{2}$ for each $h \in H - K$.

Finally, we note that μ_H is normalized on H, then $\mu_H(H) = 1$.

From these facts and from Lemma A, we have

$$d(H,G) = \int_H \mu_G(C_G(h))d\mu_H(h)$$

$$= \int_K \mu_G(C_G(h))d\mu_H(h) + \int_{H-K} \mu_G(C_G(h))d\mu_H(h)$$

$$\leq \int_K d\mu_H(h) + \int_{H-K} \mu_G(C_G(h))d\mu_H(h)$$

$$\leq \int_K d\mu_H(h) + \frac{1}{2}\int_{H-K} d\mu_H(h) = \mu_H(K) + \frac{1}{2}(\mu_H(H-K))$$

$$= \mu_H(K) + \frac{1}{2}(1-\mu_H(K)) = \frac{1}{2} + \frac{1}{2}\mu_H(K).$$

Since $d(H,G) = \frac{3}{4}$, the previous relation gives $\mu_H(K) \geq \frac{1}{2}$.

On the other hand, we may easily deduce, as in the initial argument, that

$$1 = \mu_H(H) = |H : K|\ \mu_H(K)$$

and, as before, we get $\mu_H(K) \leq \frac{1}{2}$. Therefore, $\mu_H(K) = \frac{1}{2}$ and the index $|H : K| = 2$. This means that H/K is cyclic of order 2, as claimed.

(ii). We may argue as in the previous statement (i). On a hand, we have

$$\frac{5}{8} = d(H,G) \leq \frac{1}{2} + \frac{1}{2}\mu_H(K).$$

Therefore, $\mu_H(K) \geq \frac{1}{4}$. On the other hand, the equality

$$1 = \mu_H(H) = |H : K|\ \mu_H(K)$$

gives $\mu_H(K) = \frac{1}{4}$ so that $|H : K| = 4$. This means that H/K has order 4. But, H is nonabelian, then H/K cannot be cyclic. From this, H/K is 2-elementary abelian of rank 2, as claimed. $\square$

3 An Open Question

Our Main Theorem extends (Erfanian, Lescot & Rezaei, 2007, Theorem 3.10). We are trying to extend also (Erfanian, Lescot & Rezaei, 2007, Theorems 4.3, 4.5, 4.6, 5.1, 5.3, 5.5) in the context of compact groups. However, the following open question remains unsolved both in finite and infinite case.

Open Question. *Let H be a closed subgroup of a compact group G.*

(i) *What is the structure of $H/(Z(G)\cap H)$, assuming that $d^{(n)}(H,G) = \frac{3}{4}$?*

(ii) *What is the structure of $H/(Z(G)\cap H)$, assuming that H is nonabelian and $d^{(n)}(H,G) = \frac{5}{8}$?*

The interest in Open Question is due to the fact that the commutativity degree is an invariant under isoclinism. See (Chiti, Moghaddam & Salemkar, 2005; Lescot, 1995) for details and terminology. In particular, (Chiti, Moghaddam & Salemkar, 2005, Theorem B) shows that two isoclinic finite groups have the same commutativity degree. Recently, this has been investigated in (Erfanian, Rezaei & Russo, 2010; Russo, 2007). A satisfactory answer to Open Question will allow us to classify all those isoclinic groups having an assigned n-th commutativity degree.

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